

A UNIVERSAL PROPERTY FOR GROUPOID C*-ALGEBRAS. I

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ABSTRACT. We describe representations of groupoid C*-algebras on Hilbert modules over arbitrary C*-algebras by a universal property. For Hilbert space representations, our universal property is equivalent to Renault's Integration–Disintegration Theorem. For a locally compact group, it is related to the automatic continuity of measurable group representations. It implies known descriptions of groupoid C*-algebras as crossed products for étale groupoids and transformation groupoids of group actions on spaces.

1. INTRODUCTION

The C*-algebra of a locally compact group G may be characterised uniquely up to isomorphism by a universal property: there is a natural bijection between nondegenerate *-homomorphisms $C^*(G) \rightarrow \mathcal{M}(D)$ – briefly called morphisms $C^*(G) \rightarrow D$ – and strictly continuous group homomorphisms $G \rightarrow \mathcal{U}(D)$, where $\mathcal{U}(D)$ denotes the group of unitary multipliers of a C*-algebra D .

Now let G be a locally compact, Hausdorff groupoid with a Haar system α . Renault's Integration and Disintegration Theorems describe the Hilbert space representations of the groupoid C*-algebra $C^*(G, \alpha)$ (see [14]). The Hilbert space representations alone are, however, not enough to determine a C*-algebra uniquely up to isomorphism. This complicates many arguments about groupoid C*-algebras because the details of the definition of $C^*(G, \alpha)$ reappear in every argument. We are going to describe the representations of $C^*(G, \alpha)$ on Hilbert *modules* over arbitrary C*-algebras by a universal property. This universal property determines $C^*(G, \alpha)$ uniquely up to a canonical isomorphism and should therefore simplify many arguments with groupoid C*-algebras. The groupoid C*-algebras for different Haar systems on the same groupoid are canonically Morita–Rieffel equivalent, but not isomorphic (see [9]). Hence our universal property *must* contain the Haar system. This entails some complications. To show how our universal property can be used, we apply it to two special cases, namely, étale groupoids and transformation groupoids of group actions. We describe their representations and thus their groupoid C*-algebras. This implies that the groupoid C*-algebra of an étale groupoid or a transformation groupoid for a group action is a crossed product for an inverse semigroup action or a group action, respectively. This description comes with a universal property that describes representations on Hilbert modules as well as Hilbert spaces.

Our universal property uses the commutative C*-algebras of functions on the spaces of objects, arrows, and composable pairs of arrows in G . Therefore, as it stands, it only works for Hausdorff groupoids. The non-Hausdorff case may be treated by desingularising a non-Hausdorff, locally compact groupoid to a Hausdorff,

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locally compact bigroupoid. Furthermore, there is a variant where we add Fell bundles, even non-saturated ones. The most general version of the universal property applies to non-saturated Fell bundles over bigroupoids, which we view as partial actions of bigroupoids by Hilbert bimodules (partial Morita–Rieffel equivalences). Since both Fell bundles and bigroupoids create further technical complications, we discuss them only later, in sequels to this article.

When we combine our universal property with the representation theory of commutative C^* -algebras on separable Hilbert spaces, the resulting description of representations of groupoid C^* -algebras on separable Hilbert spaces is equivalent to Renault’s Integration–Disintegration Theorem. Besides the Haar system on the groupoid, our universal property does not involve any measure theory because this would fail for representations on Hilbert *modules*. In fact, our universal property works for arbitrary (non-separable) groupoid C^* -algebras and only involves rather soft analysis. This is compensated by appropriate algebraic structures. For a locally compact group G , our universal property for Hilbert space representations gives Haar-measurable weak representations, that is, Haar-measurable maps $g \mapsto U_g$ from G to the unitary group such that $U_g U_h = U_{gh}$ holds for almost all $(g, h) \in G^2$ with respect to the Haar measure. Together with the usual universal property for group C^* -algebras, this shows that any Haar-measurable weak group representation is equal almost everywhere to a continuous group representation. Similar automatic continuity results for group representations go back to Stefan Banach and André Weil.

Throughout this article, we let G be a locally compact, Hausdorff groupoid with a Haar system, which we denote by α . Let G^0 , G^1 and G^2 be its spaces of objects, arrows and composable pairs of arrows, and let $r, s: G^1 \rightrightarrows G^0$ be its range and source maps. We recall in Section 2 how to construct C^* -correspondences between commutative C^* -algebras such as $C_0(G^i)$ for $i = 0, 1, 2$ from topological correspondences between the underlying spaces. We construct some C^* -correspondences of this type from families of measures along canonical maps $G^2 \rightarrow G^1 \rightarrow G^0$. Using these C^* -correspondences, we formulate our universal property in Section 3. We illustrate it by the example of the regular representation and relate it to the Integration and Disintegration Theorems of Renault [14]. For the universal property, we define “representations” of (G, α) on Hilbert modules. Our main theorem asserts that these representations are equivalent to representations of the groupoid C^* -algebra. We describe how to integrate and disintegrate representations in Sections 4 and 5, and we show that both constructions are inverse to each other in Section 6. This section finishes the proof of the universal property. Section 7 specialises to transformation groups and étale groupoids.

2. CONTINUOUS FAMILIES OF MEASURES AND TOPOLOGICAL CORRESPONDENCES

Our universal property is based on canonical C^* -correspondences between the commutative C^* -algebras $C_0(G^i)$ for $i = 0, 1, 2$. We are going to describe a general procedure to construct C^* -correspondences between commutative C^* -algebras. The C^* -correspondences between $C_0(G^i)$ that we need are all of this form.

A C^* -correspondence from a C^* -algebra A to another C^* -algebra D consists of a (right) Hilbert D -module with a nondegenerate $*$ -homomorphism φ from A to $\mathbb{B}(\mathcal{F})$, the C^* -algebra of adjointable operators on \mathcal{F} . We view a C^* -correspondence from A to D as an arrow $A \rightarrow D$ and usually write $A \xrightarrow{\mathcal{F}} D$. We also view φ as a *representation* of A on \mathcal{F} . Two C^* -correspondences \mathcal{F}_1 and \mathcal{F}_2 from A to D are *isomorphic* if there is a unitary bimodule map $U: \mathcal{F}_1 \xrightarrow{\sim} \mathcal{F}_2$.

We write \otimes for suitably completed tensor products of C^* -correspondences, and \odot for the tensor product of vector spaces without any completion. In particular, the

composite of two C*-correspondences $A \xrightarrow{\mathcal{E}} B$ and $B \xrightarrow{\mathcal{F}} D$ is their (balanced) tensor product $\mathcal{E} \otimes_B \mathcal{F}$, a completion of $\mathcal{E} \odot_B \mathcal{F}$.

Let X and Y be locally compact, Hausdorff spaces and let $f: X \rightarrow Y$ be a continuous map with a continuous family λ of measures λ_y along the fibres $f^{-1}(y)$ of f (see [14, Section 1]). Thus each λ_y is a positive Radon measure on X with $\text{supp}(\lambda_y) \subseteq f^{-1}(y)$. The continuity of λ means that the integration map

$$\lambda: C_c(X) \rightarrow C_c(Y), \quad \lambda(\varphi)(y) = \int_X \varphi(x) d\lambda_y(x),$$

takes values in $C_c(Y)$.

Definition 2.1. We equip $C_c(X)$ with the $C_0(X)$ - $C_0(Y)$ -bimodule structure

$$(\varphi_1 \cdot \varphi_2 \cdot \varphi_3)(x) := \varphi_1(x) \varphi_2(x) \varphi_3(f(x))$$

for $\varphi_1 \in C_0(X)$, $\varphi_2 \in C_c(X)$, $\varphi_3 \in C_0(Y)$ and with the $C_0(Y)$ -valued inner product $\langle \xi | \eta \rangle := \lambda(\overline{\xi} \cdot \eta)$, that is,

$$\langle \xi | \eta \rangle(y) := \int_{f^{-1}(y)} \overline{\xi(x)} \cdot \eta(x) d\lambda_y(x).$$

Then $C_c(X)$ is a pre-Hilbert $C_0(Y)$ -module with a nondegenerate representation of $C_0(X)$ by adjointable operators. Some nonzero $\xi \in C_c(X)$ might have $\langle \xi | \xi \rangle = 0$ unless we assume λ_y to have full support. Let $\mathcal{L}^2(X, f, \lambda)$ be the Hausdorff completion of $C_c(X)$ for this inner product, which is a C*-correspondence from $C_0(X)$ to $C_0(Y)$. In diagrams, we often briefly denote this C*-correspondence as

$$C_0(X) \xrightarrow[\lambda]{f} C_0(Y).$$

A Hilbert module over $C_0(Y)$ is the same as a continuous field of Hilbert spaces over Y . The C*-correspondence $\mathcal{L}^2(X, f, \lambda)$ is equivalent to the continuous field of Hilbert spaces with fibre $\mathcal{L}^2(f^{-1}(y), \lambda_y)$ at $y \in Y$ whose C_0 -sections are generated by $C_c(X)$; here $\xi \in C_c(X)$ is identified with the section $y \mapsto \xi|_{f^{-1}(y)}$.

Let $f: X \rightarrow Y$ and $g: Y \rightarrow Z$ be continuous maps with continuous families of measures λ and μ , respectively. Then the composite integration map

$$\mu \circ \lambda: C_c(X) \rightarrow C_c(Y) \rightarrow C_c(Z), \quad (\mu \circ \lambda)(\varphi)(z) = \int_Y \int_X \varphi(x) d\lambda_y(x) d\mu_z(y),$$

describes a continuous family of measures $\mu \circ \lambda$ along $g \circ f$.

Lemma 2.2. *The map*

$$\gamma: C_c(X) \odot C_c(Y) \rightarrow C_c(X), \quad \gamma(\varphi \otimes \psi)(x) := \varphi(x) \cdot \psi(f(x)),$$

extends uniquely to an isomorphism of $C_0(X)$ - $C_0(Z)$ -correspondences

$$\bar{\gamma}: \mathcal{L}^2(X, f, \lambda) \otimes_{C_0(Y)} \mathcal{L}^2(Y, g, \mu) \xrightarrow{\sim} \mathcal{L}^2(X, g \circ f, \mu \circ \lambda).$$

Proof. The map γ preserves the inner products defining $\mathcal{L}^2(X, f, \lambda) \otimes_{C_0(Y)} \mathcal{L}^2(Y, g, \mu)$ and $\mathcal{L}^2(X, g \circ f, \mu \circ \lambda)$ because

$$\begin{aligned} & \langle \gamma(\varphi_1 \otimes \psi_1) | \gamma(\varphi_2 \otimes \psi_2) \rangle(z) \\ &= \int_{g^{-1}(z)} \int_{f^{-1}(y)} \overline{\gamma(\varphi_1 \otimes \psi_1)(x)} \gamma(\varphi_2 \otimes \psi_2)(x) d\lambda_y(x) d\mu_z(y) \\ &= \int_{g^{-1}(z)} \int_{f^{-1}(y)} \overline{\varphi_1(x) \psi_1(y)} \varphi_2(x) \psi_2(y) d\lambda_y(x) d\mu_z(y) = \langle \psi_1 | \langle \varphi_1 | \varphi_2 \rangle \cdot \psi_2 \rangle \end{aligned}$$

for all $z \in Z$. Hence γ extends to an isometry $\bar{\gamma}: \mathcal{L}^2(X, f, \lambda) \otimes_{C_0(Y)} \mathcal{L}^2(Y, g, \mu) \rightarrow \mathcal{L}^2(X, g \circ f, \mu \circ \lambda)$. This is clearly a bimodule map. It is surjective (and hence an isomorphism of correspondences) because $\gamma: C_c(X) \odot C_c(Y) \rightarrow C_c(X)$ is already

surjective: any $\varphi' \in C_c(X)$ may be decomposed as $\varphi'(x) = \varphi(x)\psi(f(x))$ by taking $\psi \in C_c(Y)$ with $\psi(y) = 1$ for $y \in f(\text{supp}(\varphi'))$ and $\varphi := \varphi'$. \square

The compositions for measure families and C^* -correspondences are compatible by Lemma 2.2. In our brief notation, this means that the following diagram of C^* -correspondences commutes up to the canonical isomorphism γ :

$$(2.3) \quad \begin{array}{ccc} C_0(X) & \xrightarrow[\lambda]{f} & C_0(Y) \\ & \searrow \mu \circ \lambda & \downarrow \mu \circ g \\ & & C_0(Z) \end{array}$$

Definition 2.4. Let $f: X \rightarrow Y$ (“forward”) be a continuous map with a continuous family of measures λ and let $b: X \rightarrow Z$ (“backward”) be a continuous map. We define a $C_0(Z)$ -module structure on $C_c(X)$ by $(\varphi \cdot \psi)(x) := \varphi(b(x)) \cdot \psi(x)$ for $\varphi \in C_0(Z)$, $\psi \in C_c(X)$. This extends to a representation of $C_0(Z)$ on the Hilbert $C_0(Y)$ -module $\mathcal{L}^2(X, f, \lambda)$, turning it into a C^* -correspondence from $C_0(Z)$ to $C_0(X)$. We denote it by $\mathcal{L}^2(X, b, f, \lambda)$ when we want to emphasise this extra structure. We call a pair of maps $Z \xleftarrow{b} X \xrightarrow{f} Y$ with a continuous family of measures λ along f a *topological correspondence* from Z to Y .

In particular, the $C_0(X)$ - $C_0(Y)$ -correspondence constructed in Definition 2.1 from a continuous family of measures λ along a continuous map $f: X \rightarrow Y$ is associated to the topological correspondence

$$X \xleftarrow{\text{id}_X} X \xrightarrow[\lambda]{f} Y.$$

Topological correspondences are a mild generalisation of the topological quivers introduced by Muhly and Tomforde [8]: a topological quiver is a topological correspondence with the same source and target space. Basic results about topological quivers such as [8, Lemmas 6.1–4] have obvious generalisations to topological correspondences. We also get the notion of a topological correspondence if we specialise the topological correspondences between locally compact, Hausdorff groupoids introduced in [6] to locally compact spaces.

Topological correspondences may be composed by a fibre product construction, and this composition and the interior tensor product of C^* -correspondences are compatible up to a canonical isomorphism, see [8, Lemmas 6.1–4] or [6]. We give more details. Let X , Y and Z be locally compact spaces. Let (V, b_V, f_V, λ) and (W, b_W, f_W, μ) be topological correspondences from X to Y and from Y to Z , respectively. Their composite topological correspondence is the fibre product

$$V \times_{f_V, Y, b_W} W := \{(v, w) \in V \times W : f_V(v) = b_W(w)\}$$

with the maps $b := b_V \circ \text{pr}_1$ and $f := f_W \circ \text{pr}_2$, respectively, where pr_i is the projection from $V \times_{f_V, Y, b_W} W$ to the i th factor; the family of measures $\lambda \times \mu$ on $V \times_{f_V, Y, b_W} W$ is defined by

$$\int_{V \times_{f_V, Y, b_W} W} f d(\lambda \times \mu)_z := \int_W \int_V f(v, w) d\lambda_{b_W(w)}(v) d\mu_z(w)$$

for $f \in C_c(V \times_{f_V, Y, b_W} W)$ and $z \in Z$.

Proposition 2.5. *The canonical map $\gamma: C_c(V) \odot C_c(W) \rightarrow C_c(V \times_{f_V, Y, b_W} W)$, $\gamma(\varphi \otimes \psi)(v, w) := \varphi(v)\psi(w)$, extends to an isomorphism*

$$\mathcal{L}^2(V, b_V, f_V, \lambda) \otimes_{C_0(Y)} \mathcal{L}^2(W, b_W, f_W, \mu) \xrightarrow{\sim} \mathcal{L}^2(V \times_{f_V, Y, b_W} W, b, f, \lambda \times \mu)$$

of C^ -correspondences from $C_0(X)$ to $C_0(Z)$.*

Proof. A routine computation shows that γ is a bimodule map. To see that γ preserves the inner products, take $\varphi_1, \varphi_2 \in C_c(V)$ and $\psi_1, \psi_2 \in C_c(W)$. Then

$$\begin{aligned} \langle \varphi_1 \otimes \psi_1 | \varphi_2 \otimes \psi_2 \rangle(z) &= \langle \psi_1 | \langle \varphi_1 | \varphi_2 \rangle \cdot \psi_2 \rangle(z) \\ &= \int_W \int_V \overline{\psi_1(w) \varphi_1(v)} \varphi_2(v) \psi_2(w) d\lambda_{b_W(w)}(v) d\mu_z(w) \\ &= \langle \gamma(\varphi_1 \otimes \psi_1) | \gamma(\varphi_2 \otimes \psi_2) \rangle(z). \end{aligned}$$

Thus γ is an isometric bimodule map. The subspace of $C_c(V \times W)$ spanned by functions of the form $(v, w) \mapsto \varphi(v)\psi(w)$ is dense in the inductive limit topology. Hence restrictions of such functions to $V \times_{f_V, Y, b_W} W$ are linearly dense in $C_c(V \times_{f_V, Y, b_W} W)$, which is dense in $\mathcal{L}^2(V \times_{f_V, Y, b_W} W, b, f, \lambda \times \mu)$. Thus γ is surjective. \square

An *isomorphism* between two topological correspondences

$$X \xleftarrow{b_i} Z_i \xrightarrow[\lambda_i]{f_i} Y, \quad i = 1, 2,$$

is a homeomorphism $\Phi: Z_1 \xrightarrow{\sim} Z_2$ with the following properties: $f_2 \circ \Phi = f_1$, $b_2 \circ \Phi = b_1$, and the measure families λ_2 and $\Phi_*(\lambda_1)$ along $f_2: Z_2 \rightarrow Y$ are “equivalent.” This means that $\Phi_*(\lambda_1)_y$ is equivalent to $(\lambda_2)_y$ for all $y \in Y$ and that the resulting Radon–Nikodym derivatives form a continuous function on Z_2 . Here $\Phi_*(\lambda_1)$ denotes the pullback of λ_1 along Φ ; it consists of the family of pullback measures $\Phi_*(\lambda_1)_y$ for $y \in Y$. For an isomorphism as above, the formula

$$\Phi^*(g)(z) := g \circ \Phi^{-1}(z) \sqrt{\frac{d\Phi_*(\lambda_1)_{f_2(z)}(z)}{d(\lambda_2)_{f_2(z)}(z)}}$$

for $g \in C_c(Z_1)$, $z \in Z_2$ defines a unitary $C_0(X)$ - $C_0(Y)$ -bimodule isomorphism $C_c(Z_1) \rightarrow C_c(Z_2)$. Hence it extends to an isomorphism of C*-correspondences $\Phi^*: \mathcal{L}^2(Y, Z_1, \lambda_1) \xrightarrow{\sim} \mathcal{L}^2(Y, Z_2, \lambda_2)$.

Remark 2.6. The continuity assumptions on $\Phi^{\pm 1}$ and the Radon–Nikodym derivatives may be weakened. For instance, if Y is just a point, then it suffices to assume $\Phi^{\pm 1}$ to be measurable; the Radon–Nikodym derivative is automatically measurable. In general, we need measurability in the fibre directions and continuity along the base Y . We do not try to make this precise here because we shall only use continuous isomorphisms as defined above.

3. THE UNIVERSAL PROPERTY FOR GROUPOID C*-ALGEBRAS

Let α be a left-invariant Haar system on G . This is a continuous family of measures with full support along the fibres of the range map $r: G^1 \rightarrow G^0$. The right-invariant Haar system $\tilde{\alpha}$ corresponding to α is a continuous family of measures along the fibres of the source map $s: G^1 \rightarrow G^0$:

$$\begin{aligned} (\alpha f)(x) &= \int_{G^x} f(g) d\alpha^x(g), & G^x &:= \{g \in G \mid r(g) = x\}, \\ (\tilde{\alpha} f)(x) &= \int_{G_x} f(g) d\tilde{\alpha}_x(g) = \int_{G^x} f(g^{-1}) d\alpha^x(g), & G_x &:= \{g \in G \mid s(g) = x\}. \end{aligned}$$

Besides the range and source maps $r, s: G^1 \rightarrow G^0$, we shall also use the three maps $d_0, d_1, d_2: G^2 \rightarrow G^1$ defined by

$$d_0(g, h) = h, \quad d_1(g, h) = g \cdot h, \quad d_2(g, h) = g$$

for $g, h \in G^1$ with $s(g) = r(h)$. The composite maps are

$$(3.1) \quad v_0 = r \circ d_1 = r \circ d_2, \quad v_1 = r \circ d_0 = s \circ d_2, \quad v_2 = s \circ d_0 = s \circ d_1.$$

These maps are illustrated in Figure 1.

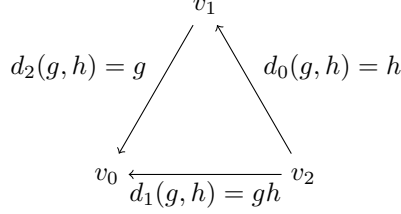


FIGURE 1. A pair $(g, h) \in G^2$ of composable arrows generates a commutative triangle of arrows in G . The edge opposite the vertex v_i is denoted by d_i for $i = 0, 1, 2$.

We use our continuous families of measures $\tilde{\alpha}$ and α along the maps s and r to produce continuous families λ_0 , λ_1 and λ_2 along the maps d_0 , d_1 and d_2 . We describe these through their integration maps $C_c(G^2) \rightarrow C_c(G^1)$:

$$\begin{aligned}
 (\lambda_0 f)(h) &= \int_{G^{r(h)}} f(g, h) d\tilde{\alpha}_{r(h)}(g), \\
 (3.2) \quad (\lambda_1 f)(k) &= \int_{G^{r(k)}} f(g, g^{-1}k) d\alpha^{r(k)}(g) = \int_{G^{s(k)}} f(kh^{-1}, h) d\tilde{\alpha}_{s(k)}(h), \\
 (\lambda_2 f)(g) &= \int_{G^{s(g)}} f(g, h) d\alpha^{s(g)}(h).
 \end{aligned}$$

The two descriptions of λ_1 use the substitution $h = g^{-1}k$, which transforms the measures as asserted because α is left invariant and $\tilde{\alpha}$ is obtained from α by the substitution $g \mapsto g^{-1}$.

The identities of maps in (3.1) also hold for the corresponding integration maps:

$$\begin{aligned}
 (\alpha \circ \lambda_1)f(x) &= \int_{G^x} \int_{G^x} f(g, g^{-1}k) d\alpha^x(g) d\alpha^x(k) \\
 &= \int_{G^x} \int_{G^x} f(g, h) d\alpha^{s(g)}(h) d\alpha^x(g) = (\alpha \circ \lambda_2)f(x), \\
 (\alpha \circ \lambda_0)f(x) &= \int_{G^x} \int_{G_x} f(g, h) d\tilde{\alpha}_x(g) d\alpha^x(h) \\
 (3.3) \quad &= \int_{G_x} \int_{G^x} f(g, h) d\alpha^x(h) d\tilde{\alpha}_x(g) = (\tilde{\alpha} \circ \lambda_2)f(x), \\
 (\tilde{\alpha} \circ \lambda_0)f(x) &= \int_{G_x} \int_{G^{r(h)}} f(g, h) d\tilde{\alpha}_{r(h)}(g) d\tilde{\alpha}_x(h) \\
 &= \int_{G_x} \int_{G_x} f(kh^{-1}, h) d\tilde{\alpha}_x(h) d\tilde{\alpha}_x(k) = (\tilde{\alpha} \circ \lambda_1)f(x).
 \end{aligned}$$

Hence we get unique continuous families of measures μ_i along $v_i: G^2 \rightarrow G^0$ for $i = 0, 1, 2$:

$$(3.4) \quad \mu_0 := \alpha \circ \lambda_1 = \alpha \circ \lambda_2, \quad \mu_1 := \alpha \circ \lambda_0 = \tilde{\alpha} \circ \lambda_2, \quad \mu_2 := \tilde{\alpha} \circ \lambda_0 = \tilde{\alpha} \circ \lambda_1.$$

For later computations, we remember the following consequence of the two formulas for λ_1 in (3.2) and the equality $\tilde{\alpha} \circ \lambda_0 = \tilde{\alpha} \circ \lambda_1$ in (3.3):

$$(3.5) \quad \iint f(g, g^{-1}k) d\alpha^{r(k)}(g) d\tilde{\alpha}_x(k) = \iint f(g, h) d\tilde{\alpha}_{r(h)}(g) d\tilde{\alpha}_x(h)$$

for all $x \in G^0$ and $f \in C_c(G^1 \times_{s, G^0, r} G_x)$.

As in Definition 2.1, we assign C*-correspondences to all the families of measures above. The resulting diagram of C*-correspondences in Figure 2 commutes up to canonical isomorphisms of C*-correspondences by (2.3) and (3.4).

$$\begin{array}{ccccc}
 C_0(G^1) & \xrightarrow{s} & C_0(G^0) & & \\
 \downarrow \alpha \downarrow r & \swarrow \tilde{\alpha} & \uparrow v_2 \uparrow \mu_2 & \nwarrow \tilde{\alpha} & \\
 C_0(G^0) & \xleftarrow{\mu_1} & C_0(G^2) & \xrightarrow{d_1} & C_0(G^1) \\
 & \swarrow v_1 & \downarrow \lambda_2 \downarrow d_2 & \searrow v_0 & \downarrow \alpha \downarrow r \\
 & & C_0(G^1) & \xrightarrow{r} & C_0(G^0)
 \end{array}$$

FIGURE 2. Canonical isomorphisms of the C*-correspondences associated to the continuous families of measures along the maps $G^2 \rightarrow G^1 \rightarrow G^0$. For each triangle, (3.4) and Lemma 2.2 give a canonical isomorphism between the two C*-correspondences that form the boundary of the triangle.

Let D be a C*-algebra and \mathcal{F} a Hilbert D -module. Our main theorem (Theorem 3.14 below) says that representations of the groupoid C*-algebra $C^*(G, \alpha)$ on \mathcal{F} are equivalent to representations of (G, α) on \mathcal{F} as in the following definition:

Definition 3.6. A *representation* of (G, α) on \mathcal{F} is a pair (φ, U) , where $\varphi: C_0(G^0) \rightarrow \mathbb{B}(\mathcal{F})$ is a representation – which turns \mathcal{F} into a C*-correspondence from $C_0(G^0)$ to D – and U is an isomorphism of C*-correspondences from $C_0(G^1)$ to D that makes the diagram of C*-correspondences

$$(3.7) \quad \begin{array}{ccc}
 C_0(G^1) & \xrightarrow{s} & C_0(G^0) \\
 \downarrow \alpha \downarrow r & \swarrow U & \downarrow \mathcal{F} \\
 C_0(G^0) & \xrightarrow{\mathcal{F}} & D
 \end{array}$$

commute and that is such that the two composite isomorphisms

$$\mathcal{L}^2(G^2, v_2, \mu_2) \otimes_{\varphi} \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(G^2, v_0, \mu_0) \otimes_{\varphi} \mathcal{F}$$

in Figure 3 are the same.

More explicitly, (3.7) says that U is a unitary operator

$$U: \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F}$$

that intertwines the left actions of $C_0(G^1)$ on these Hilbert D -modules. The right diagram in Figure 3 is based on the isomorphism of C*-correspondences

$$\begin{aligned}
 1 \otimes U: \mathcal{L}^2(G^2, d_1, \lambda_1) \otimes_{C_0(G^1)} (\mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F}) \\
 \xrightarrow{\sim} \mathcal{L}^2(G^2, d_1, \lambda_1) \otimes_{C_0(G^1)} (\mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F}).
 \end{aligned}$$

The canonical isomorphisms in Figure 2 allow us to turn this into an isomorphism of C*-correspondences

$$(3.8) \quad \mathcal{L}^2(G^2, v_2, \mu_2) \otimes_{\varphi} \mathcal{F} \xrightarrow[\cong]{d_1^*(U)} \mathcal{L}^2(G^2, v_0, \mu_0) \otimes_{\varphi} \mathcal{F}.$$

The left diagram in Figure 3 contains the two isomorphisms of C*-correspondences

$$(3.9) \quad \mathcal{L}^2(G^2, v_2, \mu_2) \otimes_{\varphi} \mathcal{F} \xrightarrow[\cong]{d_0^*(U)} \mathcal{L}^2(G^2, v_1, \mu_1) \otimes_{\varphi} \mathcal{F} \xrightarrow[\cong]{d_2^*(U)} \mathcal{L}^2(G^2, v_0, \mu_0) \otimes_{\varphi} \mathcal{F},$$

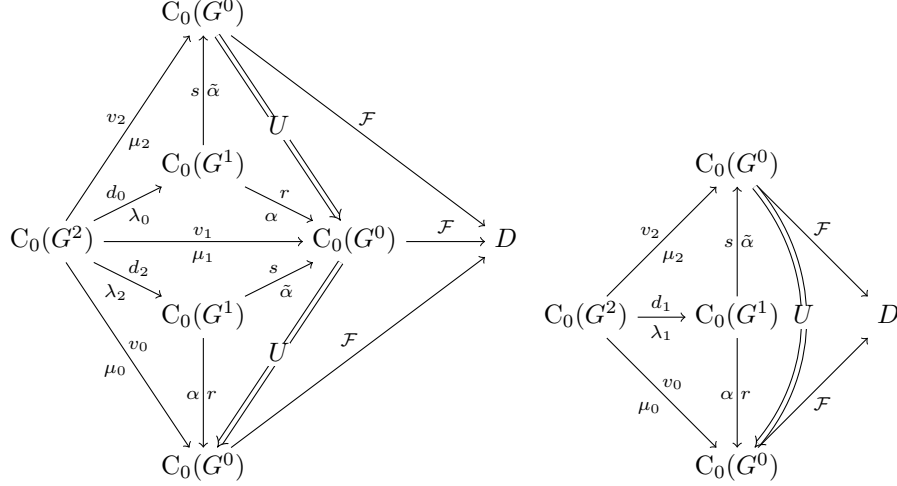


FIGURE 3. Two parallel isomorphisms of correspondences constructed from U . Each triangle or square corresponds to one isomorphism of C^* -correspondences. The unlabelled ones involve the canonical isomorphisms of C^* -correspondences in Figure 2.

where $d_0^*(U)$ and $d_2^*(U)$ are built out of U in the same way as $d_1^*(U)$. The last condition in Definition 3.6 means that

$$(3.10) \quad d_1^*(U) = d_2^*(U) \circ d_0^*(U).$$

3.1. The regular representation. We may combine the left regular representations of $C^*(G, \alpha)$ on the Hilbert spaces $\mathcal{L}^2(G_x, \tilde{\alpha}_x)$ for $x \in G^0$ into a single representation on the Hilbert $C_0(G^0)$ -module $\mathcal{F} := \mathcal{L}^2(G^1, s, \tilde{\alpha})$. To illustrate our definition, we describe the corresponding representation of (G, α) on \mathcal{F} . We equip \mathcal{F} with the left action φ of $C_0(G^0)$ defined so that $\varphi(f)$ for $f \in C_0(G^0)$ acts by point-wise multiplication with the function $f \circ r \in C_b(G^1)$. Thus the C^* -correspondence $\mathcal{F}: C_0(G^0) \rightarrow C_0(G^0)$ comes from the topological correspondence

$$G^0 \xleftarrow{r} G^1 \xrightarrow{s, \tilde{\alpha}} G^0.$$

The tensor product $\mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F}$ is the C^* -correspondence from $C_0(G^1)$ to $C_0(G^0)$ associated to the composite of the underlying topological correspondences. This involves the fibre product space $G^1 \times_{s, G^0, r} G^1 = G^2$ and the maps $d_2: G^2 \rightarrow G^1$, $(g, h) \mapsto g$, and $v_2: G^2 \rightarrow G^0$, $(g, h) \mapsto s(h)$. The measure family along v_2 for this composite is, by definition, $\tilde{\alpha} \circ \lambda_0 = \mu_2$. That is, $\mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F}$ comes from the topological correspondence

$$(3.11) \quad G^1 \xleftarrow{d_0} G^2 \xrightarrow[\mu_2]{v_2} G^0.$$

Similarly, the other tensor product $\mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F}$ comes from the topological correspondence

$$(3.12) \quad G^1 \xleftarrow{\text{pr}_1} G^1 \times_{r, G^0, r} G^1 \xrightarrow[\tilde{\alpha} \circ \alpha]{s \circ \text{pr}_2} G^0$$

with $\tilde{\alpha} \circ \alpha(f)(x) := \iint f(g, k) d\alpha^{r(k)}(g) d\tilde{\alpha}_x(k)$ for $f \in C_c(G^1 \times_{r, G^0, r} G^1)$, $x \in G^0$. We claim that the topological correspondences (3.11) and (3.12) are isomorphic through the homeomorphism

$$(3.13) \quad \Upsilon: G^2 = G^1 \times_{s, G^0, r} G^1 \xrightarrow{\sim} G^1 \times_{r, G^0, r} G^1, \quad (g, h) \mapsto (g, g \cdot h).$$

The conditions $\text{pr}_1 \circ \Upsilon = d_0$ and $s \circ \text{pr}_2 \circ \Upsilon = v_2$ are trivial. Equation (3.5) says that $\Upsilon_*(\mu_2) = \tilde{\alpha} \circ \alpha$. Thus Υ is an isomorphism of topological correspondences. It induces an isomorphism of C*-correspondences

$$U: \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F}.$$

We claim that (φ, U) is a representation of (G, α) .

The C*-correspondence $\mathcal{L}^2(G^2, v_i, \mu_i) \otimes_{\varphi} \mathcal{F}$ from $C_0(G^2)$ to $C_0(G^0)$ is associated to the topological correspondence

$$G^2 \xleftarrow{\text{pr}_1} G^2 \times_{v_i, G^0, r} G^1 \xrightarrow[\tilde{\alpha} \circ \mu_i]{s \circ \text{pr}_2} G^0.$$

The isomorphisms $d_i^*(U)$ in (3.8) and (3.9) come from isomorphisms of topological correspondences by construction. These are the homeomorphisms

$$\begin{aligned} d_1^*(\Upsilon): G^2 \times_{v_2, G^0, r} G^1 &\rightarrow G^2 \times_{v_0, G^0, r} G^1, & (g, h, l) &\mapsto (g, h, gh), \\ d_0^*(\Upsilon): G^2 \times_{v_2, G^0, r} G^1 &\rightarrow G^2 \times_{v_1, G^0, r} G^1, & (g, h, l) &\mapsto (g, h, hl), \\ d_2^*(\Upsilon): G^2 \times_{v_1, G^0, r} G^1 &\rightarrow G^2 \times_{v_0, G^0, r} G^1, & (g, h, l) &\mapsto (g, h, gl). \end{aligned}$$

Since the multiplication in G is associative, these isomorphisms of topological correspondences satisfy $d_2^*(\Upsilon) \circ d_0^*(\Upsilon) = d_1^*(\Upsilon)$. Therefore, $d_2^*(U) \circ d_0^*(U) = d_1^*(U)$. Hence (φ, U) is a representation of (G, α) .

3.2. Formulation of the universal property. The groupoid C*-algebra $C^*(G, \alpha)$ is defined in [13, Chapter II].

Theorem 3.14. *Let G be a locally compact, Hausdorff groupoid with a Haar system α . Let D be a C*-algebra and \mathcal{F} a Hilbert D -module. There is a bijection between representations of $C^*(G, \alpha)$ on \mathcal{F} and representations of (G, α) on \mathcal{F} . It is natural in the following two ways:*

- (1) *Let \mathcal{F}_1 and \mathcal{F}_2 be two Hilbert D -modules and let $V: \mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ be an isometry. Then it intertwines two representations of $C^*(G, \alpha)$ on \mathcal{F}_1 and \mathcal{F}_2 if and only if V intertwines the corresponding representations of (G, α) .*
- (2) *Let \mathcal{E} be a C*-correspondence from D to a C*-algebra D' . A representation of $C^*(G, \alpha)$ or of (G, α) on \mathcal{F} induces a representation of $C^*(G, \alpha)$ or of (G, α) on $\mathcal{F} \otimes_D \mathcal{E}$, respectively. The bijections between representations of $C^*(G, \alpha)$ and (G, α) on \mathcal{F} and $\mathcal{F} \otimes_D \mathcal{E}$ are compatible with these induction processes.*

The two naturality properties above are inspired by an analogous definition for representations of *-algebras by unbounded operators in [7, Section 3]. The same argument as there shows that the universal property characterises $C^*(G, \alpha)$ uniquely up to a canonical isomorphism. So Theorem 3.14 gives an alternative definition of $C^*(G, \alpha)$. This proof uses 3.14.(1) only for unitary operators V .

Before we prove Theorem 3.14, we relate it to Renault's Integration and Disintegration Theorems in [14]. Thus we specialise to the case where $D = \mathbb{C}$ and \mathcal{F} is a separable Hilbert space. The representation theory of commutative C*-algebras on separable Hilbert spaces is very well known (see, for instance, [3, Sections 8.2–3]). When we apply it to the representation φ of $C_0(G^0)$, we get a measure class $[\nu]$ on G^0 , a $[\nu]$ -measurable field of Hilbert spaces $\mathcal{H} = (\mathcal{H}_x)_{x \in G^0}$ on G^0 , and a unitary operator $\mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(G^0, \nu, \mathcal{H})$ that intertwines φ and the representation of $C_0(G^0)$ on $\mathcal{L}^2(G^0, \nu, \mathcal{H})$ by pointwise multiplication.

The tensor products of $\mathcal{L}^2(G^0, \nu, \mathcal{H})$ with the C*-correspondences $\mathcal{L}^2(G^1, s, \tilde{\alpha})$ and $\mathcal{L}^2(G^1, r, \alpha)$ are representations of $C_0(G^1)$, which may be described similarly. We may compute the measure classes and measurable fields on G^1 directly. For $\mathcal{L}^2(G^1, s, \tilde{\alpha})$, we get the measure class $[\nu \circ \tilde{\alpha}]$ and the pull-back field $s^*(\mathcal{H})$ with

fibre $\mathcal{H}_{s(g)}$ at $g \in G^1$. For $\mathcal{L}^2(G^1, r, \alpha)$, we get the measure class $[\nu \circ \alpha]$ and the pull-back field $r^*(\mathcal{H})$ with fibre $\mathcal{H}_{r(g)}$ at $g \in G^1$. A unitary intertwiner U between these two representations of $C_0(G^1)$ can only exist if the measure classes $[\nu \circ \tilde{\alpha}]$ and $[\nu \circ \alpha]$ on G^1 are equal. By definition, this means that the measure ν on G^0 is *quasi-invariant*. The description of representations of $C_0(G^1)$ through a measure class and a measurable field of Hilbert spaces is natural in the formal sense. That is, any unitary intertwiner between the two representations of $C_0(G^1)$ on $\mathcal{L}^2(G^1, \nu \circ \alpha, s^*\mathcal{H})$ and $\mathcal{L}^2(G^1, \nu \circ \tilde{\alpha}, r^*\mathcal{H})$ must have the form

$$(3.15) \quad (Uf)(g) = U_g(f(g)) \cdot \sqrt{\frac{d\nu \circ \alpha}{d\nu \circ \tilde{\alpha}}}$$

for $f \in \mathcal{L}^2(G^1, \nu \circ \alpha, s^*\mathcal{H})$ and almost all $g \in G^1$, where $(U_g)_{g \in G^1}$ is an isomorphism between the measurable fields of Hilbert spaces $s^*\mathcal{H}$ and $r^*\mathcal{H}$. That is, U_g is a unitary operator $U_g: \mathcal{H}_{s(g)} \xrightarrow{\sim} \mathcal{H}_{r(g)}$ for all $g \in G^1$ outside a null set for the measure class $[\nu \circ \alpha]$. The Radon–Nikodym derivative in (3.15) is well defined because $[\nu \circ \tilde{\alpha}] = [\nu \circ \alpha]$. Conversely, any measurable family of unitary operators $U_g: \mathcal{H}_{s(g)} \xrightarrow{\sim} \mathcal{H}_{r(g)}$ defines a unitary intertwiner between the two representations of $C_0(G^1)$ on $\mathcal{L}^2(G^1, \nu \circ \alpha, s^*\mathcal{H})$ and $\mathcal{L}^2(G^1, \nu \circ \tilde{\alpha}, r^*\mathcal{H})$.

Similarly, we may identify

$$\mathcal{L}^2(G^2, \mu_j, v_j) \otimes_{C_0(G^0)} \mathcal{L}^2(G^0, \nu, \mathcal{H}) \cong \mathcal{L}^2(G^2, \nu \circ \mu_j, v_j^*\mathcal{H});$$

that is, we take \mathcal{L}^2 -sections of the measurable fields of Hilbert spaces with fibres $\mathcal{H}_{r(g)}$, $\mathcal{H}_{s(g)} = \mathcal{H}_{r(h)}$ and $\mathcal{H}_{s(h)}$ at $(g, h) \in G^2$ with respect to the measures $\nu \circ \mu_j$ for $j = 0, 1, 2$, respectively. The isomorphisms of C^* -correspondences $d_i^*(U)$ for $i = 0, 1, 2$ become isomorphisms

$$\begin{aligned} d_0^*(U): \mathcal{L}^2(G^2, \mu_2, v_2) &\xrightarrow{\sim} \mathcal{L}^2(G^2, \mu_1, v_1), & d_0^*(U) &\sim (U_h)_{(g,h) \in G^2}, \\ d_1^*(U): \mathcal{L}^2(G^2, \mu_2, v_2) &\xrightarrow{\sim} \mathcal{L}^2(G^2, \mu_0, v_0), & d_1^*(U) &\sim (U_{gh})_{(g,h) \in G^2}, \\ d_2^*(U): \mathcal{L}^2(G^2, \mu_1, v_1) &\xrightarrow{\sim} \mathcal{L}^2(G^2, \mu_0, v_0), & d_2^*(U) &\sim (U_g)_{(g,h) \in G^2}, \end{aligned}$$

where \sim means that the unitary $d_0^*(U)$ corresponds to the measurable family of unitary operators $U_h: \mathcal{H}_{s(h)} \xrightarrow{\sim} \mathcal{H}_{r(h)}$, and so on, as in (3.15). Equation (3.10) holds if and only if $U_{gh} = U_g \circ U_h$ for almost all $(g, h) \in G^2$ with respect to the measure class $[\nu \circ \mu_0] = [\nu \circ \mu_1] = [\nu \circ \mu_2]$. The Radon–Nikodym derivatives in (3.15) cancel automatically.

Summing up, the representation theory of commutative C^* -algebras translates a representation (φ, U) of (G, α) on a separable Hilbert space in Definition 3.6 into

- a measure class $[\nu]$ on G^0 that is quasi-invariant, that is, $[\nu \circ \alpha] = [\nu \circ \tilde{\alpha}]$;
- a $[\nu]$ -measurable field of Hilbert spaces \mathcal{H} on G^0 ;
- unitary operators $U_g: \mathcal{H}_{s(g)} \xrightarrow{\sim} \mathcal{H}_{r(g)}$ for $[\nu \circ \alpha]$ -almost all $g \in G^1$, which satisfy $U_{gh} = U_g \circ U_h$ for $[\nu \circ \mu_i]$ -almost all $(g, h) \in G^2$.

Comparing this with Renault’s notion of a representation in [14, Definition 3.4], there is only one technical difference about which null sets are allowed in G^1 and G^2 . In [14], there is one $[\nu]$ -negligible subset N of G^0 such that the unitaries U_g are defined and satisfy $U_{gh} = U_g \circ U_h$ whenever $(g, h) \in G^2$ and $r(g), s(g) = r(h), s(h) \notin N$. This extra information is often useful, but it is not needed to integrate a representation. The formulas for integration of representations in [14] still work if we allow arbitrary null sets in G^1 and G^2 . If G^1 is second countable, then [12, Lemma 3.3] allows to modify (U_g) on a set of measure 0 so that there is a null set N as above.

What are the Hilbert space representations of (G, α) if G is a group? Since G^0 has only one point, the quasi-invariant measure on G^0 is irrelevant and the measurable field over G^0 is simply the Hilbert space \mathcal{H} on which the representation takes place. The isomorphism of correspondences U is a unitary intertwiner for the pointwise multiplication action of $C_0(G)$ on $L^2(G, \mathcal{H}) \cong L^2(G) \otimes \mathcal{H}$. This is equivalent to a measurable family of unitary operators $U_g \in \mathcal{U}(\mathcal{H})$. The condition $d_2^*(U) \circ d_0^*(U) = d_1^*(U)$ holds if and only if $U_g \circ U_h = U_{g \cdot h}$ for almost all $(g, h) \in G^2$. Thus $(U_g)_{g \in G}$ is a measurable weak representation of G on \mathcal{H} (compare [12] for the notation of weak representations). The usual universal property of $C^*(G)$ uses *continuous* representations. Both universal properties together say that any measurable weak representation is equal almost everywhere to a continuous group representation. This result is rather easy to prove. First, any measurable weak representation of G integrates to a nondegenerate representation of the convolution algebra $L^1(G)$. Secondly, any nondegenerate Banach $L^1(G)$ -module comes from a continuous representation of G because the regular representation on $L^1(G)$ is continuous. Third, this continuous representation must be equal almost everywhere to the given weak representation in order to integrate to the same representation of $L^1(G)$.

Remark 3.16. Our universal property also works for non-separable Hilbert spaces. But unitary intertwiners on $L^2(G, \mathcal{H})$ are no longer equivalent to measurable families of unitary operators on \mathcal{H} up to equality almost everywhere. For instance, consider the family of unitary operators U_t on $\ell^2(\mathbb{R})$, where $U_t(\delta_t) := \delta_{t+1}$, $U_t(\delta_{t+1}) := \delta_t$, and $U_t(\delta_s) := \delta_s$ for $s \in \mathbb{R} \setminus \{t, t+1\}$. This family describes the identity operator on $L^2(\mathbb{R}, \ell^2 \mathbb{R})$ because if $s \in \mathbb{R}$, then $U_t(\delta_s) = \delta_s$ for almost all $t \in \mathbb{R}$. Nevertheless, the set of $t \in \mathbb{R}$ with $U_t = 1$ is empty.

What happens if G is a locally compact space viewed as a groupoid? In this case, $s = r$ and $\alpha = \tilde{\alpha}$. Equation (3.10) says that $U \cdot U = U$. Since U is unitary, we may cancel U here, so U is the identity map. Thus a representation in the sense of Definition 3.6 is simply a representation of $C_0(G^0)$, which is also the groupoid C*-algebra. So Theorem 3.14 is trivial in this case. In contrast, the Integration and Disintegration Theorems in [14] are non-trivial even for spaces viewed as groupoids. Since there is no general disintegration theory for representations of commutative C*-algebras on Hilbert modules, the theory in [14] is limited to Hilbert space representations.

The proof of Theorem 3.14 requires two constructions. Integration takes a representation of (G, α) to one of $C^*(G, \alpha)$, and disintegration takes a representation of $C^*(G, \alpha)$ to one of (G, α) . We shall discuss these two constructions in Sections 4 and 5. We prove that they are inverse to each other in Section 6.

4. INTEGRATION

Let (φ, U) be a representation of (G, α) on a Hilbert module \mathcal{F} over a C*-algebra D . We are going to “integrate” it to a representation of $C^*(G, \alpha)$.

Notation 4.1 (Creation operators). Let \mathcal{E} be a Hilbert module over a C*-algebra B , let \mathcal{F} be a B - D -correspondence, and $x \in \mathcal{E}$. Let $T_x: \mathcal{F} \rightarrow \mathcal{E} \otimes_B \mathcal{F}$ denote the operator $y \mapsto x \otimes y$. Its adjoint T_x^* maps $z \otimes y \mapsto \langle x|z \rangle \cdot y$ for $x, z \in \mathcal{E}$, $y \in \mathcal{F}$.

Let $f \in C_c(G^1)$. Choose functions $h_1, h_2 \in C_c(G^1)$ with $h_1(g) = h_2(g) = 1$ for all $g \in \text{supp } f$. Let $L(f) \in \mathbb{B}(\mathcal{F})$ be the composite operator

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{T_{h_2}} \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F} & \xrightarrow{M_f \otimes \text{id}_{\mathcal{F}}} \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F} \\ & \cong \downarrow U & \cong \downarrow U \\ & \mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F} & \xrightarrow{M_f \otimes \text{id}_{\mathcal{F}}} \mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F} \xrightarrow{T_{h_1}^*} \mathcal{F}. \end{array}$$

Here we view h_1 and h_2 as elements of $\mathcal{L}^2(G^1, r, \alpha)$ and $\mathcal{L}^2(G^1, s, \tilde{\alpha})$, respectively, and M_f denotes the operator of pointwise multiplication by f , which is how $C_0(G^1)$ acts in the C^* -correspondence $\mathcal{L}^2(G^1, s, \tilde{\alpha})$. The square commutes because U , as an isomorphism of C^* -correspondences, intertwines the left actions of $C_0(G^1)$. We will write M_f instead of $M_f \otimes \text{id}_{\mathcal{F}}$ in the following to simplify notation.

Lemma 4.2. *The operator $L(f)$ does not depend on h_1 and h_2 and satisfies*

$$\|L(f)\| \leq \|\alpha(|f|)\|_{\infty}^{1/2} \cdot \|\tilde{\alpha}(|f|)\|_{\infty}^{1/2} \leq \max\{\|\alpha(|f|)\|_{\infty}, \|\tilde{\alpha}(|f|)\|_{\infty}\} = \|f\|_I$$

for all $f \in C_c(G^1)$. Here α and $\tilde{\alpha}$ also denote the integration maps

$$\alpha(|f|)(x) := \int_{G^x} |f(g)| d\alpha^x(g), \quad \tilde{\alpha}(|f|)(x) := \int_{G_x} |f(g)| d\tilde{\alpha}_x(g).$$

The I -norm $\|f\|_I$ is defined as the maximum of $\|\alpha(|f|)\|_{\infty}$ and $\|\tilde{\alpha}(|f|)\|_{\infty}$.

Proof. There are $f_1, f_2 \in C_c(G^1)$ with $\text{supp}(f_i) \subseteq \text{supp}(f)$ and $f(g) = \overline{f_1(g)} \cdot f_2(g)$ for all $g \in G^1$. A good choice for later estimates is to take $f_2(g) := \sqrt{|f(g)|}$ and $f_1(g) := \overline{f(g)}/f_2(g)$ if $f(g) \neq 0$ and $f_1(g) := 0$ if $f(g) = 0$. Now we use $M_{f_i} \circ T_{h_i} = T_{f_i \cdot h_i} = T_{f_i}$ and that $M_{\overline{f_1}} = M_{f_1}^*$ commutes with U to simplify

$$\langle \xi | L(f) \eta \rangle = \langle h_1 \otimes \xi | U(\overline{f_1} f_2 h_2 \otimes \eta) \rangle = \langle f_1 h_1 \otimes \xi | U(f_2 h_2 \otimes \eta) \rangle = \langle f_1 \otimes \xi | U(f_2 \otimes \eta) \rangle$$

for all $\xi, \eta \in \mathcal{F}$. That is,

$$(4.3) \quad L(\overline{f_1} \cdot f_2) = T_{f_1}^* U T_{f_2}$$

for all $f_1, f_2 \in C_c(G^1)$, where \cdot denotes the pointwise product. This does not depend on h_1 and h_2 any more. Moreover, $\|T_f\| = \|T_f^*\| = \|f\|$ and $\|U\| = 1$ give

$$\|L(f)\| \leq \|f_1\|_{\mathcal{L}^2(G^1, r, \alpha)} \|f_2\|_{\mathcal{L}^2(G^1, s, \tilde{\alpha})}.$$

If we choose f_1 and f_2 as indicated above, then $|f_1(g)|^2 = |f_2(g)|^2 = |f(g)|$ for all $g \in G^1$, so that $\|f_1\|_{\mathcal{L}^2(G^1, r, \alpha)}^2 = \|\alpha(|f|)\|_{\infty}$ and $\|f_2\|_{\mathcal{L}^2(G^1, s, \tilde{\alpha})}^2 = \|\tilde{\alpha}(|f|)\|_{\infty}$. This gives the desired norm estimate for $L(f)$. \square

The previous lemma implies that the map L is continuous from $C_c(G^1)$ with the inductive limit topology to $\mathbb{B}(\mathcal{F})$ with the norm topology because the inductive limit topology is stronger than the I -norm topology.

Lemma 4.4. *The linear span of $L(f)\xi$ for $f \in C_c(G^1)$, $\xi \in \mathcal{F}$ is dense in \mathcal{F} .*

Proof. The linear span of $T_{f_2}\eta$ for $f_2 \in C_c(G^1)$, $\eta \in \mathcal{F}$ is dense in $\mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F}$. Since U is unitary, the linear span of $U T_{f_2}\eta$ for such f_2 and η is still dense in $\mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F}$. Then the linear span of $L(\overline{f_1} \cdot f_2)(\eta) = T_{f_1}^* U T_{f_2}\eta$ for $f_1, f_2 \in C_c(G^1)$, $\eta \in \mathcal{F}$ is dense in \mathcal{F} because the Hilbert $C_0(G^0)$ -module $\mathcal{L}^2(G^1, r, \alpha)$ is full and φ is nondegenerate. Here we have used (4.3). \square

Proposition 4.5. *The map L is a nondegenerate $*$ -homomorphism from the convolution algebra $(C_c(G^1), *)$ to $\mathbb{B}(\mathcal{F})$.*

Proof. We are going to prove below that

$$(4.6) \quad L(f_1)^* L(f_2) = L(f_1^* * f_2).$$

for all $f_1, f_2 \in C_c(G^1)$. We first claim that this implies $L(f)^* = L(f^*)$ and then $L(f_1)L(f_2) = L(f_1 * f_2)$ for $f, f_1, f_2 \in C_c(G^1)$; that is, L is a $*$ -representation. Since $f^{**} = f$ for all $f \in C_c(G^1)$, (4.6) is equivalent to $L(f_1^*)^* L(f_2) = L(f_1 * f_2)$. Since $C_c(G^1)$ is a $*$ -algebra, this implies

$$\begin{aligned} L(f_1^*)^* L(f_2^*)^* L(f_3) &= L(f_1^*)^* L(f_2 * f_3) = L(f_1 * (f_2 * f_3)) = L((f_1 * f_2) * f_3) \\ &= L((f_1 * f_2)^*)^* L(f_3) = L(f_2^* * f_1^*)^* L(f_3) = L(f_1^*)^* L(f_2) L(f_3). \end{aligned}$$

Hence $\langle L(f_1^*)\xi_1 | L(f_2^*)^* L(f_3)\xi_2 \rangle = \langle L(f_1^*)\xi_1 | L(f_2)L(f_3)\xi_2 \rangle$ for all $\xi_1, \xi_2 \in \mathcal{F}$. Vectors of the form $L(f_1^*)\xi_1$ or $L(f_3)\xi_2$ span a dense subspace of \mathcal{F} by Lemma 4.4. Hence $L(f_2^*)^* = L(f_2)$ for all $f_2 \in C_c(G^1)$ and L is nondegenerate.

It remains to prove (4.6). Our analysis in the Hilbert space case suggests that we need (3.10). We will use the equivalent formula $d_2^*(U)^* d_1^*(U) = d_0^*(U)$ with the operators $d_i^*(U)$ for $i = 0, 1, 2$ in (3.8) and (3.9) because $d_1^*(U) = d_2^*(U) \circ d_0^*(U)$ would lead to a proof that $L(f_1 * f_2) = L(f_1)L(f_2)$. We are given $f_1, f_2 \in C_c(G^1)$ and define $f \in C_c(G^2)$ by

$$f(g_1, g_2) = \overline{f_1(g_1)} f_2(g_1 \cdot g_2).$$

Furthermore, let $h' \in C_c(G^2)$ be some function with $h' \cdot f = f$, that is, h' is 1 on the (compact) support of f . Equation (3.10) implies

$$T_{h'}^* M_f d_0^*(U) T_{h'} = T_{h'}^* M_f d_2^*(U)^* d_1^*(U) T_{h'} = T_{h'}^* d_2^*(U)^* M_f d_1^*(U) T_{h'},$$

where we used that $d_2^*(U)^*$ commutes with $C_0(G^2)$. We are going to prove that

$$(4.7) \quad T_{h'}^* M_f d_0^*(U) T_{h'} = L(f_1^* * f_2), \quad T_{h'}^* d_2^*(U)^* M_f d_1^*(U) T_{h'} = L(f_1)^* L(f_2).$$

This will finish the proof of (4.6).

We begin with some preparatory observations. Our proof depends on the isomorphisms of C*-correspondences in Figure 2 such as

$$(4.8) \quad \mathcal{L}^2(G^2, d_0, \lambda_0) \otimes_{C_0(G^1)} \mathcal{L}^2(G^1, r, \alpha) \cong \mathcal{L}^2(G^2, v_1, \mu_1).$$

This isomorphism is described in Lemma 2.2 and maps $\varphi_1 \otimes \varphi_2$ for $\varphi_1 \in C_c(G^2)$, $\varphi_2 \in C_c(G^1)$ to the function $(g_1, g_2) \mapsto \varphi_1(g_1, g_2) \cdot \varphi_2(d_0(g_1, g_2)) = \varphi_1(g_1, g_2) \cdot \varphi_2(g_2)$. Hence the inverse isomorphism can be taken to send $\varphi_1 \mapsto \varphi_1 \otimes k$, where $k \in C_c(G^1)$ is such that $k(g_2) = 1$ for all $(g_1, g_2) \in \text{supp } \varphi_1$. Similar remarks apply to all commutative triangles in Figure 2.

The operators $T_{h'}^*$ and $T_{h'}$ in (4.7) have slightly different meanings: the first treats $h' \in \mathcal{L}^2(G^2, v_1, \mu_1)$, the second as $h' \in \mathcal{L}^2(G^2, v_2, \mu_2)$; this is implicit in (4.7) because $M_f d_0^*(U)$ and $d_2^*(U)^* M_f d_1^*(U)$ are operators from $\mathcal{L}^2(G^2, v_2, \mu_2)$ to $\mathcal{L}^2(G^2, v_1, \mu_1)$. We write $(T_{h'}^{v_1})^*$ or $T_{h'}^{v_2}$ to clarify in which C*-correspondence we view h' . We shall also need $T_{h'}^{d_0}$, and so on. The definition of $d_i^*(U)$ through $\text{id}_{\mathcal{L}^2(G^2, d_i, \lambda_i)} \otimes U$ implies

$$(4.9) \quad d_i^*(U) T_{h'}^{d_i} = T_{h'}^{d_i} U \quad \text{for } i = 0, 1, 2.$$

Now we compute the operator $(T_{h'}^{v_1})^* M_f d_0^*(U) T_{h'}^{v_2}$. First, we rewrite $h' = h' \otimes k$, where k is 1 on a sufficiently large compact subset, compare the discussion after (4.8). Then $T_{h'}^{v_2} = T_{h'}^{d_0} T_k^s$ and $T_{h'}^{v_1} = T_{h'}^{d_0} T_k^r$. Using (4.9) as well, we get

$$(T_{h'}^{v_1})^* M_f d_0^*(U) T_{h'}^{v_2} = (T_k^r)^* (T_{h'}^{d_0})^* M_f T_{h'}^{d_0} U T_k^s.$$

The operator $(T_{h'}^{d_0})^* M_f T_{h'}^{d_0}$ does the following on $\varphi \otimes \xi$ for $\varphi \in \mathcal{L}^2(G^1, r, \alpha)$, $\xi \in \mathcal{F}$:

$$\varphi \otimes \xi \xrightarrow{T_{h'}^{d_0}} h' \otimes \varphi \otimes \xi \xrightarrow{M_f} f h' \otimes \varphi \otimes \xi \xrightarrow{(T_{h'}^{d_0})^*} \langle h' | f h' \rangle_{\mathcal{L}^2(G^2, d_0, \lambda_0)} \cdot \varphi \otimes \xi,$$

where the product in $\langle h' | fh' \rangle \cdot \varphi$ is the pointwise multiplication action of $C_0(G^1)$ on $\varphi \in \mathcal{L}^2(G^1, r, \alpha)$. Since $h' = 1$ on the support of f , we get $(T_{h'}^{d_0})^* M_f T_{h'}^{d_0} = M_{\lambda_0(f)}$ with

$$\begin{aligned} \lambda_0(f)(g_2) &= \langle h' | fh' \rangle_{\mathcal{L}^2(G^2, d_0, \lambda_0)}(g_2) = \int_{G^{r(g_2)}} f(g_1, g_2) d\tilde{\alpha}_{r(g_2)}(g_1) \\ &= \int_{G^{r(g_2)}} f(g_1^{-1}, g_2) d\alpha^{r(g_2)}(g_1) = \int_{G^{r(g_2)}} \overline{f_1(g_1^{-1})} f_2(g_1^{-1} \cdot g_2) d\alpha^{r(g_2)}(g_1) \\ &= (f_1^* * f_2)(g_2). \end{aligned}$$

This proves the first half of (4.7).

Now we prove the second half. We rewrite $T_{h'}^{v_2} = T_{h'}^{d_1} T_k^s$ and $T_{h'}^{v_1} = T_{h'}^{d_2} T_k^s$ because $v_2 = s \circ d_1$ and $v_1 = s \circ d_2$. We use (4.9) to compute

$$(T_{h'}^{v_1})^* d_2^*(U)^* M_f d_1^*(U) T_{h'}^{v_2} = (T_k^s)^* U^* (T_{h'}^{d_2})^* M_f T_{h'}^{d_1} U T_k^s.$$

Let $\varphi \in C_c(G^1) \subseteq \mathcal{L}^2(G^1, r, \alpha)$ and $\xi \in \mathcal{F}$. Then $M_f T_{h'}^{d_1}$ first maps $\varphi \otimes \xi$ to $f \cdot h' \otimes \varphi \otimes \xi \in \mathcal{L}^2(G^2, d_1, \lambda_1) \otimes_{C_0(G^1)} \mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F}$. To apply the operator $(T_{h'}^{d_2})^*$ to this, we need the canonical isomorphism

$$\begin{aligned} \mathcal{L}^2(G^2, d_1, \lambda_1) \otimes_{C_0(G^1)} \mathcal{L}^2(G^1, r, \alpha) \\ \cong \mathcal{L}^2(G^2, v_0, \mu_0) \cong \mathcal{L}^2(G^2, d_2, \lambda_2) \otimes_{C_0(G^1)} \mathcal{L}^2(G^1, r, \alpha) \end{aligned}$$

from Figure 2 and then take the inner product with h' in the first tensor factor. Since h' is 1 wherever our functions are supported, we have $fh' = f$, and the inner product with h' simply applies the integration map λ_2 . Thus $(T_{h'}^{d_2})^* M_f T_{h'}^{d_1}$ maps $\varphi \otimes \xi$ to $\psi \otimes \xi$ with

$$\begin{aligned} \psi(g_1) &= \int_{G^{s(g_1)}} f(g_1, g_2) (\varphi \circ d_1)(g_1, g_2) d\alpha^{s(g_1)}(g_2) \\ &= \int_{G^{s(g_1)}} \overline{f_1(g_1)} f_2(g_1 g_2) \varphi(g_1 g_2) d\alpha^{s(g_1)}(g_2) \\ &= \overline{f_1(g_1)} \int_{G^{s(g_1)}} f_2(g_2) \varphi(g_2) d\alpha^{r(g_1)}(g_2). \end{aligned}$$

As a consequence,

$$(T_{h'}^{d_2})^* M_f T_{h'}^{d_1} = M_{f_1}^* T_k^r (T_k^r)^* M_{f_2}.$$

Putting things together gives

$$(T_{h'}^{v_1})^* (M_f d_2^*(U)^* d_1^*(U)) T_{h'}^{v_2} = (T_k^s)^* U^* M_{f_1}^* T_k^r (T_k^r)^* M_{f_2} U T_k^s = L(f_1)^* * L(f_2). \quad \square$$

Since the $*$ -representation L of $(C_c(G^1), *)$ is bounded in the I -norm, it extends uniquely to a representation of $C^*(G, \alpha)$, which we still denote by L . This is the integrated form of the representation (φ, U) .

5. DISINTEGRATION

In this section, we construct a representation (φ, U) of (G, α) from a representation of $C^*(G, \alpha)$. Actually, we shall start with a more technical setting, allowing densely defined representations of $C_c(G^1)$. Renault's Disintegration Theorem also applies in this generality, and several results need such representations.

Definition 5.1. Let \mathcal{F} be a Hilbert module over a C^* -algebra D and let \mathcal{F}_0 be a vector space with a linear map $\iota: \mathcal{F}_0 \rightarrow \mathcal{F}$ with dense image. Let $\text{Hom}(\mathcal{F}_0, \mathcal{F})$, be the vector space of all linear maps $\mathcal{F}_0 \rightarrow \mathcal{F}$. A *pre-representation* of $C_c(G^1)$ on \mathcal{F} is a linear map $L: C_c(G^1) \rightarrow \text{Hom}(\mathcal{F}_0, \mathcal{F})$ such that

- (1) for all $\xi, \eta \in \mathcal{F}_0$, the map $f \mapsto \langle \iota(\xi) | L(f)\eta \rangle_D$ from $C_c(G^1)$ to D is continuous in the inductive limit topology on $C_c(G^1)$ and the norm topology on D ;

- (2) $\langle L(f_1)\xi | L(f_2)\eta \rangle_D = \langle \iota(\xi) | L(f_1^* * f_2)\eta \rangle_D$ for all $\xi, \eta \in \mathcal{F}_0$;
- (3) the linear span of $L(f)\xi$ for $f \in C_c(G^1)$, $\xi \in \mathcal{F}_0$ is dense in \mathcal{F} .

We do not need the map ι to be injective. The assumptions in Definition 5.1 imply, however, that $L(f)\xi = 0$ for all $f \in C_c(G^1)$, $\xi \in \mathcal{F}_0$ with $\iota(\xi) = 0$ because $\langle L(f)\xi | L(f_2)\xi_2 \rangle = 0$ for all $f_2 \in C_c(G^1)$, $\xi_2 \in \mathcal{F}_0$ by (2) and linear combinations of $L(f_2)\xi_2$ are dense in \mathcal{F} by (3). So it would be no loss of generality to assume ι to be injective: we may replace $\iota: \mathcal{F}_0 \rightarrow \mathcal{F}$ by the injective linear map $\tilde{\iota}: \tilde{\mathcal{F}}_0 := \mathcal{F}_0 / \ker(\iota) \rightarrow \mathcal{F}$, $\tilde{\iota}(\xi + \ker(\iota)) := \iota(\xi)$, and L by $\tilde{L}: C_c(G^1) \rightarrow \text{Hom}(\tilde{\mathcal{F}}_0, \mathcal{F})$, $\tilde{L}(f)(\xi + \ker(\iota)) := L(f)\xi$.

Throughout this section, we fix a groupoid G with a Haar system α and a pre-representation $(L, \mathcal{F}_0, \iota)$ of $C_c(G^1)$. Disintegration will produce a representation (φ, U) of (G, α) on \mathcal{F} . First, we construct the representation φ of $C_0(G^0)$ on \mathcal{F} . Define $r^*(f_0) \cdot f_1 \in C_c(G^1)$ for $f_0 \in C_b(G^0)$, $f_1 \in C_c(G^1)$ by $r^*(f_0) \cdot f_1(g) := f_0(r(g)) \cdot f_1(g)$ for all $g \in G^1$. Then define

$$\varphi_0: C_b(G^0) \odot C_c(G^1) \odot \mathcal{F}_0 \rightarrow \mathcal{F}, \quad f_0 \otimes f_1 \otimes \xi \mapsto L(r^*(f_0) \cdot f_1)\xi.$$

Lemma 5.2. *There is a unique representation $\varphi: C_0(G^0) \rightarrow \mathbb{B}(\mathcal{F})$ with*

$$\varphi(f_0)(L(f_1)\xi) = \varphi_0(f_0 \otimes f_1 \otimes \xi)$$

for all $f_0 \in C_0(G^0)$, $f_1 \in C_c(G^1)$, $\xi \in \mathcal{F}_0$.

Proof. Let $f_0 \in C_b(G^1)$. Define $f'_0 \in C_b(G^1)$ by $f'_0(g) := \sqrt{\|f_0\|_\infty^2 - |f_0(g)|^2}$ for all $g \in G^1$, so that $f_0^* f_0 + (f'_0)^* f'_0 = \|f_0\|_\infty^2 \cdot 1$ in $C_b(G^1)$. If $f_1 \in C_c(G^1)$, $\xi \in \mathcal{F}_0$, then

$$\begin{aligned} & \langle \varphi_0(f_0 \otimes f_1 \otimes \xi) | \varphi_0(f_0 \otimes f_1 \otimes \xi) \rangle \\ & \leq \langle \varphi_0(f_0 \otimes f_1 \otimes \xi) | \varphi_0(f_0 \otimes f_1 \otimes \xi) \rangle + \langle \varphi_0(f'_0 \otimes f_1 \otimes \xi) | \varphi_0(f'_0 \otimes f_1 \otimes \xi) \rangle \\ & = \langle L(r^*(f_0)f_1)\xi | L(r^*(f_0)f_1)\xi \rangle + \langle L(r^*(f'_0)f_1)\xi | L(r^*(f'_0)f_1)\xi \rangle \\ & = \langle \iota(\xi) | L(f_1^* * (r^*(f_0^* f_0 + (f'_0)^* f'_0))f_1) \rangle \\ & = \|f_0\|^2 \langle \iota(\xi) | L(f_1^* * f_1) \rangle = \|f_0\|^2 \langle L(f_1)\xi | L(f_1)\xi \rangle. \end{aligned}$$

Since $L(C_c(G^1))(\mathcal{F}_0)$ is linearly dense in \mathcal{F} , there is a unique bounded linear operator $\varphi(f_0): \mathcal{F} \rightarrow \mathcal{F}$ with

$$\varphi(f_0)(L(f_1)\xi) = \varphi_0(f_0 \otimes f_1 \otimes \xi) = L(r^*(f_0)f_1)\xi$$

for all $f_1 \in C_c(G^1)$, $\xi \in \mathcal{F}_0$. The operator $\varphi(f_0)$ is adjointable with adjoint $\varphi(f_0^*)$ because $f_1^* * (r^*(f_0)f'_1) = (r^*(f_0^*)f_1)^* * f'_1$ in $C_c(G^1)$ for all $f_1, f'_1 \in C_c(G^1)$. The map $\varphi: C_b(G^1) \rightarrow \mathbb{B}(\mathcal{F})$ is linear and multiplicative because the action of $C_b(G^1)$ on $C_c(G^1)$ by pointwise multiplication is a module structure. The restriction of φ to $C_0(G^1)$ is nondegenerate because $C_0(G^1) \cdot C_c(G^1) = C_c(G^1)$ and $L(C_c(G^1))(\mathcal{F}_0)$ is linearly dense in \mathcal{F} . \square

Next we are going to construct linear maps with dense range

$$\begin{aligned} \tau_s: C_c(G^2) \odot \mathcal{F}_0 &\rightarrow \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_\varphi \mathcal{F}, \\ \tau_r: C_c(G^2) \odot \mathcal{F}_0 &\rightarrow \mathcal{L}^2(G^1, r, \alpha) \otimes_\varphi \mathcal{F}, \end{aligned}$$

such that

$$\langle \tau_s(x) | \tau_s(x) \rangle = \langle \tau_r(x) | \tau_r(x) \rangle$$

for all $x \in C_c(G^2) \odot \mathcal{F}_0$. Hence there is a unique unitary operator

$$U: \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_\varphi \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(G^1, r, \alpha) \otimes_\varphi \mathcal{F}$$

with $U(\tau_s(x)) = \tau_r(x)$ for all $x \in C_c(G^2) \odot \mathcal{F}_0$. We will check later that (φ, U) is a representation of (G, α) . The construction of τ_s, τ_r is a special case of the following lemma.

Lemma 5.3. *Let $p: X \rightarrow G^0$ be a continuous map and let λ be a continuous family of measures along p . Then the map $\tau: C_c(X) \odot C_c(G^1) \odot \mathcal{F}_0 \rightarrow \mathcal{L}^2(X, p, \lambda) \otimes_\varphi \mathcal{F}$, $f_0 \otimes f_1 \otimes \xi \mapsto f_0 \otimes L(f_1)\xi$, extends uniquely to a linear map*

$$\bar{\tau}: C_c(X \times_{p, G^0, r} G^1) \odot \mathcal{F}_0 \rightarrow \mathcal{L}^2(X, p, \lambda) \otimes_\varphi \mathcal{F},$$

such that $f \mapsto \bar{\tau}(f \otimes \xi)$ for $f \in C_c(X \times_{p, G^0, r} G^1)$ is continuous in the inductive limit topology for all $\xi \in \mathcal{F}_0$. Furthermore, if $F_1, F_2 \in C_c(X \times_{p, G^0, r} G^1)$, $\xi_1, \xi_2 \in \mathcal{F}_0$, then

$$(5.4) \quad \langle \bar{\tau}(F_1 \otimes \xi_1) | \bar{\tau}(F_2 \otimes \xi_2) \rangle = \langle \iota(\xi_1) | L(\langle F_1 | F_2 \rangle) \xi_2 \rangle,$$

where $\langle F_1 | F_2 \rangle \in C_c(G^1)$ is defined by

$$(5.5) \quad \begin{aligned} \langle F_1 | F_2 \rangle(g) &:= \iint \overline{F_1(x, h^{-1})} F_2(x, h^{-1}g) d\lambda^{s(h)}(x) d\alpha^{r(g)}(h) \\ &= \iint \overline{F_1(x, h)} F_2(x, hg) d\lambda^{r(h)}(x) d\tilde{\alpha}_{r(g)}(h). \end{aligned}$$

Proof. Let $f_1, f_3 \in C_c(X)$, $f_2, f_4 \in C_c(G^1)$, $\xi_1, \xi_2 \in \mathcal{F}_0$. We compute

$$\begin{aligned} \langle \tau(f_1 \otimes f_2 \otimes \xi_1) | \tau(f_3 \otimes f_4 \otimes \xi_2) \rangle &= \langle L(f_2)\xi_1 | \varphi(\langle f_1 | f_3 \rangle) L(f_4)\xi_2 \rangle \\ &= \langle \iota(\xi_1) | L(f_2^* * (r^*(\langle f_1 | f_3 \rangle) f_4)) \xi_2 \rangle. \end{aligned}$$

Here $f_2^* * (r^*(\langle f_1 | f_3 \rangle) f_4) \in C_c(G^1)$ is given by

$$\begin{aligned} f_2^* * (r^*(\langle f_1 | f_3 \rangle) f_4)(g) &= \int_{G^1} f_2^*(h) (r^*(\langle f_1 | f_3 \rangle) f_4)(h^{-1}g) d\alpha^{r(g)}(h) \\ &= \int_{G^1} \int_X \overline{f_2(h^{-1}) f_1(x)} f_3(x) f_4(h^{-1}g) d\lambda^{r(h^{-1}g)}(x) d\alpha^{r(g)}(h) \\ &= \int_{G^1} \int_X \overline{f_2(h) f_1(x)} f_3(x) f_4(hg) d\lambda^{r(h)}(x) d\tilde{\alpha}_{r(g)}(h). \end{aligned}$$

This is the right hand side in (5.5) if we let $F_1(x, g) := f_1(x)f_2(g)$ and $F_2(x, g) := f_3(x)f_4(g)$. Thus (5.4) with $\bar{\tau} = \tau$ holds for F_1, F_2 in the image of $C_c(X) \odot C_c(G^1)$ in $C_c(X \times_{p, G^0, r} G^1)$.

Let $F \in C_c(X \times_{p, G^0, r} G^1)$ and $\xi \in \mathcal{F}_0$. Let V and W be open, relatively compact subsets of X and G^1 , respectively, so that $V \times_{p, G^0, r} W$ is a neighbourhood of the support of F , that is, $F \in C_0(V \times_{p, G^0, r} W)$. The linear span of functions of the form $f_1 \otimes f_2$ with $f_1 \in C_0(V) \subseteq C_c(X)$, $f_2 \in C_0(W) \subseteq C_c(G^1)$ is dense in the Banach space $C_0(V \times_{p, G^0, r} W) \subseteq C_c(X \times_{p, G^0, r} G^1)$ by the Stone–Weierstraß Theorem. Hence there is a sequence $(F_n)_{n \in \mathbb{N}}$ in $C_0(V) \odot C_0(W)$ whose image in $C_0(V \times_{p, G^0, r} W)$ converges towards F . We claim that $\tau(F_n \otimes \xi)$ is a Cauchy sequence in \mathcal{F} . Then we are going to define $\bar{\tau}(F \otimes \xi) = \lim \tau(F_n \otimes \xi)$.

We may use (5.4) to compute $\langle \tau(F_n \otimes \xi) | \tau(F_m \otimes \xi) \rangle$ for all $n, m \in \mathbb{N}$ because it holds for elementary tensors. The continuity assumption for L in Definition 5.1 shows that this is a continuous bilinear map in the two entries F_n, F_m . Therefore,

$$\lim_{n, m \rightarrow \infty} \langle \tau(F_n \otimes \xi) | \tau(F_m \otimes \xi) \rangle = \langle \iota(\xi) | L(\langle F | F \rangle) \xi \rangle$$

with $\langle F | F \rangle \in C_c(G^1)$ as in (5.5). Hence

$$\begin{aligned} \|\tau(F_n \otimes \xi) - \tau(F_m \otimes \xi)\|^2 &= \langle \tau(F_n \otimes \xi) | \tau(F_n \otimes \xi) \rangle - \langle \tau(F_n \otimes \xi) | \tau(F_m \otimes \xi) \rangle \\ &\quad - \langle \tau(F_m \otimes \xi) | \tau(F_n \otimes \xi) \rangle + \langle \tau(F_m \otimes \xi) | \tau(F_m \otimes \xi) \rangle \end{aligned}$$

converges to 0 for $n, m \rightarrow \infty$. Thus $\tau(F_n \otimes \xi)$ is a Cauchy sequence in $\mathcal{L}^2(X, p, \lambda) \otimes_\varphi \mathcal{F}$. We let $\bar{\tau}(F \otimes \xi)$ be its limit. This does not depend on the choice of V, W and (F_n) because mixing two sequences of this type gives a Cauchy sequence as well, by the same argument. The map $(F, \xi) \mapsto \bar{\tau}(F \otimes \xi)$ is bilinear and hence

extends to a linear map $\bar{\tau}: C_c(X \times_{p,G^0,r} G^1) \odot \mathcal{F}_0 \rightarrow \mathcal{L}^2(X, p, \lambda) \otimes_{\varphi} \mathcal{F}$. It has dense image because already τ has dense image. The formula (5.4) holds everywhere by continuity and because it holds for F_1, F_2 in the dense subspace $C_c(X) \odot C_c(G^1)$ of $C_c(X \times_{p,G^0,r} G^1)$. \square

We apply Lemma 5.3 to the maps $r, s: G^1 \rightrightarrows G^0$ with the measure families $\alpha, \tilde{\alpha}$. This gives linear maps with dense image

$$\begin{aligned}\tau_s: C_c(G^1 \times_{s,G^0,r} G^1) \odot \mathcal{F}_0 &\rightarrow \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F}, \\ \tau_r: C_c(G^1 \times_{r,G^0,r} G^1) \odot \mathcal{F}_0 &\rightarrow \mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F}.\end{aligned}$$

We identify $G^1 \times_{s,G^0,r} G^1 = G^2$ with $G^1 \times_{r,G^0,r} G^1$ through the homeomorphisms $(g, h) \mapsto (g, g^{\pm 1}h)$ going back and forth. Let $F_1, F_2 \in C_c(G^2)$, $\xi_1, \xi_2 \in \mathcal{F}_0$. Then

$$\begin{aligned}\langle \tau_s(F_1 \otimes \xi_1) | \tau_s(F_2 \otimes \xi_2) \rangle &= \langle \iota(\xi_1) | L(\langle F_1 | F_2 \rangle_s) \xi_2 \rangle, \\ \langle \tau_r(F_1 \otimes \xi_1) | \tau_r(F_2 \otimes \xi_2) \rangle &= \langle \iota(\xi_1) | L(\langle F_1 | F_2 \rangle_r) \xi_2 \rangle\end{aligned}$$

with

$$\begin{aligned}\langle F_1 | F_2 \rangle_s(k) &= \iint \overline{F_1(x, h, xh)} F_2(x, hk, xhk) d\tilde{\alpha}_{r(h)}(x) d\tilde{\alpha}_{r(k)}(h), \\ \langle F_1 | F_2 \rangle_r(k) &= \iint \overline{F_1(x, x^{-1}h, h)} F_2(x, x^{-1}hk, hk) d\alpha^{r(h)}(x) d\tilde{\alpha}_{r(k)}(h)\end{aligned}$$

by (5.5). Here we described points in G^2 through $x, h, y \in G^1$ with $xh = y$ to clarify the identification of $G^1 \times_{r,G^0,r} G^1$ with G^2 . Equation (3.5) applied to the function

$$f^k(g, h) := \overline{F_1(g, h)} F_2(g, hk)$$

in $C_c(G^1 \times_{s,G^0,r} G_{r(k)})$ for fixed $k \in G^1$ implies $\langle F_1 | F_2 \rangle_s(k) = \langle F_1 | F_2 \rangle_r(k)$ for all $k \in G^1$. Hence τ_s and τ_r induce the same inner product on $C_c(G^2) \odot \mathcal{F}_0$. So there is a unique unitary

$$U: \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F}$$

with $U\tau_s(x) = \tau_r(x)$ for all $x \in C_c(G^2) \otimes \mathcal{F}_0$. More precisely, $U\tau_s(x) = \tau_r(v(x))$, where $v: C_c(G^1 \times_{s,r} G^1) \odot \mathcal{F}_0 \xrightarrow{\sim} C_c(G^1 \times_{r,r} G^1) \odot \mathcal{F}_0$ is the isomorphism induced by the homeomorphism $\Upsilon: G^1 \times_{s,r} G^1 \xrightarrow{\sim} G^1 \times_{r,r} G^1$, $(g, h) \mapsto (g, gh)$ in (3.13).

Proposition 5.6. *The pair (φ, U) associated to a pre-representation of $C_c(G^1)$ is a representation of (G, α) .*

Proof. First we check that U is an isomorphism of correspondences. That is, it is a $C_0(G^1)$ -module homomorphism for the canonical left $C_0(G^1)$ -module structures by pointwise multiplication on the first tensor factors $\mathcal{L}^2(G^1, s, \tilde{\alpha})$ and $\mathcal{L}^2(G^1, r, \alpha)$. The maps τ_s and τ_r are $C_0(G^1)$ -module homomorphisms if we let $f_1 \in C_0(G^1)$ act on $f_2 \in C_c(G^1 \times_{s,r} G^1) \odot \mathcal{F}_0$ and $f'_2 \in C_c(G^1 \times_{r,r} G^1) \odot \mathcal{F}_0$ by

$$(f_1 \cdot f_2)(g, h) := f_1(g)f_2(g, h), \quad (f_1 \cdot f'_2)(g, k) := f_1(g)f'_2(g, k).$$

These give the same $C_0(G^1)$ -module structure on $C_c(G^2) \odot \mathcal{F}_0$. So the unitary U is a $C_0(G^1)$ -module homomorphism.

The unitaries $d_i^*(U)$ in (3.8) and (3.9) act between $\mathcal{L}^2(G^2, v_j, \mu_j) \otimes_{\varphi} \mathcal{F}$ for $j \in \{0, 1, 2\} \setminus \{i\}$. Lemma 5.3 provides a linear map with dense range

$$\tau_{v_j}: C_c(G^2 \times_{v_j,G^0,r} G^1) \odot \mathcal{F}_0 \rightarrow \mathcal{L}^2(G^2, v_j, \mu_j) \otimes_{\varphi} \mathcal{F}.$$

The elements of $G^2 \times_{v_j,G^0,r} G^1$ are configurations of three arrows (g, h, x) with $(g, h) \in G^2$ and $r(x) = r(g)$ for $j = 0$, $r(x) = s(g) = r(h)$ for $j = 1$, and $r(x) = s(h)$ for $j = 2$, respectively. In particular, $G^2 \times_{v_2,G^0,r} G^1$ is the space G^3 of triples of composable arrows (g, h, x) in G .

When we define $d_0^*(U)$ as in (3.9), we identify $G^2 \times_{v_2, G^0, r} G^1 \cong G^1 \times_{s, G^0, r \circ \text{pr}_1} (G^1 \times_{s, r} G^1)$ and $G^2 \times_{v_1, G^0, r} G^1 \cong G^1 \times_{s, G^0, r \circ \text{pr}_1} (G^1 \times_{r, r} G^1)$ and use that U composes functions with the homeomorphism $G^1 \times_{r, r} G^1 \rightarrow G^1 \times_{s, r} G^1$, $(h, k) \mapsto (h, h^{-1}k)$. Thus $d_0^*(U)$ extends the operator

$$C_c(G^2 \times_{v_2, G^0, r} G^1) \odot \mathcal{F}_0 \rightarrow C_c(G^2 \times_{v_1, G^0, r} G^1) \odot \mathcal{F}_0, \quad f \otimes \xi \mapsto (f \circ \Upsilon_0) \otimes \xi,$$

with $\Upsilon_0(g, h, x) := (g, h, h^{-1}x)$. Similarly, $d_2^*(U)$ extends the operator

$$C_c(G^2 \times_{v_1, G^0, r} G^1) \odot \mathcal{F}_0 \rightarrow C_c(G^2 \times_{v_0, G^0, r} G^1) \odot \mathcal{F}_0, \quad f \otimes \xi \mapsto (f \circ \Upsilon_2) \otimes \xi,$$

with $\Upsilon_2(g, h, x) := (g, h, g^{-1}x)$. And $d_1^*(U)$ extends the operator

$$C_c(G^2 \times_{v_2, G^0, r} G^1) \odot \mathcal{F}_0 \rightarrow C_c(G^2 \times_{v_0, G^0, r} G^1) \odot \mathcal{F}_0, \quad f \otimes \xi \mapsto (f \circ \Upsilon_1) \otimes \xi,$$

with $\Upsilon_1(g, h, x) := (g, h, (gh)^{-1}x)$. The equation $d_2^*(U)d_0^*(U) = d_1^*(U)$ follows from $\Upsilon_1 = \Upsilon_0 \circ \Upsilon_2$, which is the associativity of the multiplication in G . \square

6. INTEGRATION VERSUS DISINTEGRATION

This section finishes the proof of Theorem 3.14. Let \mathcal{F} be a Hilbert module over a C^* -algebra D . Given a representation of (G, α) on \mathcal{F} , we have constructed a representation of the convolution algebra $C_c(G^1)$ bounded with respect to the I -norm in Section 4; this extends to a representation of $C^*(G, \alpha)$. Conversely, given a representation of $C^*(G, \alpha)$ or merely a pre-representation of $C_c(G^1)$ as in Definition 5.1, we have constructed a representation of (G, α) in Section 5. First we prove that these two constructions are inverse to each other. Hence we get the asserted bijection between representations of (G, α) and $C^*(G, \alpha)$. Then we check that the bijection has the two naturality properties in Theorem 3.14.

Proposition 6.1. *Let $(L, \mathcal{F}_0, \iota)$ be a pre-representation of $C_c(G^1)$ on \mathcal{F} . Let (φ, U) be its disintegration. Then the integrated form L' of (φ, U) satisfies $L'(f)(\iota(\xi)) = L(f)(\xi)$ for all $f \in C_c(G^1)$, $\xi \in \mathcal{F}_0$, and $L'(f)(L(f_2)\xi) = L(f * f_2)(\xi)$ for all $f, f_2 \in C_c(G^1)$, $\xi \in \mathcal{F}_0$.*

Proof. Let $f_1, f_2, f_3 \in C_c(G^1)$ and $\xi_1, \xi_2 \in \mathcal{F}_0$. We compute the inner product $\langle f_3 \otimes \iota(\xi_1) | f_1 \otimes L(f_2)\xi_2 \rangle$ in $\mathcal{L}^2(G^1, r, \alpha) \otimes_\varphi \mathcal{F}$:

$$\begin{aligned} \langle f_3 \otimes \iota(\xi_1) | f_1 \otimes L(f_2)\xi_2 \rangle &= \langle \iota(\xi_1) | \varphi(\langle f_3 | f_1 \rangle) L(f_2)\xi_2 \rangle \\ &= \langle \iota(\xi_1) | L(r^*(\langle f_3 | f_1 \rangle) \cdot f_2)\xi_2 \rangle, \end{aligned}$$

where $r^*(\langle f_3 | f_1 \rangle) \cdot f_2 \in C_c(G^1)$ is given by

$$r^*(\langle f_3 | f_1 \rangle) \cdot f_2(g) = \int_{G^1} \overline{f_3(x)} f_1(x) f_2(g) d\alpha^{r(g)}(x)$$

Thus $T_{f_3}^*(f_1 \otimes L(f_2)\xi_2) = L(r^*(\langle f_3 | f_1 \rangle) \cdot f_2)\xi_2$; the annihilation operator $T_{f_3}^*$ is the adjoint of the creation operator T_{f_3} as in Notation 4.1. By construction, the unitary U maps $f_1 \otimes L(f_2)\xi_2$ onto $\tau(F \otimes \xi_2)$ for $F \in C_c(G^1 \times_{r, G^0, r} G^1)$ defined by $F(x, g) := f_1(x) f_2(x^{-1}g)$. Since the image of $C_c(G^1) \odot C_c(G^1)$ in $C_c(G^2)$ is dense in the inductive limit topology, the computation above implies that $T_{f_3}^* U(f_1 \otimes L(f_2)\xi_2) = L(\psi)\xi_2$ with

$$\psi = \int \overline{f_3(x)} F(x, g) d\alpha^{r(g)}(x) = \int \overline{f_3(x)} f_1(x) f_2(x^{-1}g) d\alpha^{r(g)}(x) = (\overline{f_3} f_1) * f_2(g).$$

Summing up,

$$T_{f_3}^* U T_{f_1}(L(f_2)\xi) = T_{f_3}^* U(f_1 \otimes L(f_2)\xi) = L((\overline{f_3} f_1) * f_2)\xi$$

for all $f_1, f_2, f_3 \in C_c(G^1)$, $\xi \in \mathcal{F}_0$.

Equation (4.3) describes the integrated form L' of (φ, U) as $L'(\overline{f_3}f_1) := T_{f_3}^* U T_{f_1}$. So the computation above shows that $L'(f)(L(f_2)\xi) = L(f * f_2)\xi$ for all $f, f_2 \in C_c(G^1)$, $\xi \in \mathcal{F}_0$. Since L' is a $*$ -representation of the $*$ -algebra $C_c(G^1)$ on \mathcal{F} ,

$$\begin{aligned} \langle L'(f)\iota(\xi_1)|L(f_2)\xi_2 \rangle &= \langle \iota(\xi_1)|L'(f^*)L(f_2)\xi_2 \rangle \\ &= \langle \iota(\xi_1)|L(f^* * f_2)\xi_2 \rangle = \langle L(f)\xi_1|L(f_2)\xi_2 \rangle \end{aligned}$$

for all $f, f_2 \in C_c(G^1)$, $\xi_1, \xi_2 \in \mathcal{F}_0$. Since vectors of the form $L(f_2)\xi_2$ span a dense subspace in \mathcal{F} , this implies $L'(f)\iota(\xi_1) = L(f)\xi_1$. \square

Corollary 6.2. *For any pre-representation $(L, \mathcal{F}_0, \iota)$ of $C_c(G^1)$ there is a representation $L': C_c(G^1) \rightarrow \mathbb{B}(\mathcal{F})$ that is bounded with respect to the I -norm and satisfies $L'(f)(\iota(\xi)) = L(f)(\xi)$ for all $f \in C_c(G^1)$, $\xi \in \mathcal{F}_0$. In particular, a representation of $C_c(G^1)$ is I -norm bounded if and only if it is continuous in the inductive limit topology.* \square

Proof of Theorem 3.14. A representation of $C^*(G, \alpha)$ on \mathcal{F} is equivalent to a nondegenerate $*$ -representation of the $*$ -algebra $C_c(G^1)$ that is bounded with respect to the I -norm; by Corollary 6.2, this is equivalent to continuity in the inductive limit topology. When we disintegrate such a representation L to a representation (φ, U) of (G, α) and integrate (φ, U) to a representation of $C_c(G^1)$, we get back the original representation L by Proposition 6.1.

Now we start with a representation (φ, U) of (G, α) and integrate it to a representation L of $C_c(G^1)$. Let (φ', U') be the representation of (G, α) obtained by disintegrating L . We claim that $\varphi' = \varphi$ and $U' = U$. Let $f_1, f_2 \in C_c(G^1)$, $f_0 \in C_c(G^0)$ and $\xi \in \mathcal{F}$. Then $f_1 r^*(f_0) \otimes \xi = f_1 \otimes \varphi(f_0)\xi$ in $\mathcal{L}^2(G^1, r, \alpha) \otimes_\varphi \mathcal{F}$. Therefore, $T_{r^*(f_0) \cdot f_1}^* = \varphi(f_0^*)T_{f_1}^*$ in $\mathbb{B}(\mathcal{L}^2(G^1, r, \alpha) \otimes_\varphi \mathcal{F}, \mathcal{F})$. So

$$\varphi(f_0)L(\overline{f_1}f_2)\xi = \varphi(f_0)T_{f_1}^* U T_{f_2}\xi = T_{r^*(f_0) \cdot f_1}^* U T_{f_2}\xi = L(r^*(f_0)\overline{f_1}f_2)\xi.$$

Thus $\varphi' = \varphi$ by the definition of φ' in Lemma 5.2.

Equation (4.3) implies

$$\langle f_1 \otimes \xi_1 | U(f_2 \otimes \xi_2) \rangle = \langle \xi_1 | T_{f_1}^* U T_{f_2} \xi_2 \rangle = \langle \xi_1 | L(\overline{f_1}f_2) \xi_2 \rangle$$

for all $f_1, f_2 \in C_c(G^1)$ and $\xi_1, \xi_2 \in \mathcal{F}$. This shows that the representation L uniquely determines U because the inner products $\langle f_1 \otimes \xi_1 | U(f_2 \otimes \xi_2) \rangle$ determine U . Both (φ, U) and (φ', U') integrate to the same representation L by Proposition 6.1. This implies $U = U'$ because L determines U uniquely.

Let \mathcal{F}_1 and \mathcal{F}_2 be Hilbert D -modules with representations (φ_1, U_1) and (φ_2, U_2) of (G, α) and let L_1 and L_2 be the corresponding representations of $C_c(G^1)$. Let $J: \mathcal{F}_1 \hookrightarrow \mathcal{F}_2$ be an isometry. This intertwines (φ_1, U_1) and (φ_2, U_2) if and only if $J\varphi_1(f) = \varphi_2(f)J$ for all $f \in C_0(G^0)$ and $(\text{id}_{\mathcal{L}^2(G^1, r, \alpha)} \otimes J)U_1 = U_2(\text{id}_{\mathcal{L}^2(G^1, s, \tilde{\alpha})} \otimes J)$. Then

$$L_2(f)J = T_h^* U_2 T_f J = T_h^* U_2 (\text{id} \otimes J) T_f = T_h^* (\text{id} \otimes J) U_1 T_f = J T_h^* U_1 T_f = J L_1(f),$$

that is, J intertwines the representations L_1 and L_2 of $C_c(G^1)$. Then it also intertwines the unique extensions of L_1 and L_2 to $C^*(G, \alpha)$.

Conversely, assume that J intertwines two representations L_1 and L_2 of $C_c(G^1)$ on \mathcal{F}_1 and \mathcal{F}_2 . Then J also intertwines the representations φ_i of $C_0(G^0)$ defined in Lemma 5.2 and the maps $C_c(G^2) \odot \mathcal{F}_i \rightarrow \mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi_i} \mathcal{F}_i$ and $C_c(G^2) \odot \mathcal{F}_i \rightarrow \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi_i} \mathcal{F}_i$ used to construct U ; that is, the constructions in Lemmas 5.2 and Lemma 5.3 are natural in the formal sense with respect to isometric intertwiners J . Therefore, J intertwines the representations (φ_1, U_1) and (φ_2, U_2) obtained by disintegrating L_1 and L_2 .

The previous two paragraphs establish that the bijections between representations of (G, α) and $C^*(G, \alpha)$ on Hilbert D -modules have the property (1) in Theorem 3.14. Property (2) in Theorem 3.14 is obvious from our construction of the integrated form of a representation of (G, α) . \square

Corollary 6.3. *There is a universal representation (φ^u, U^u) of (G, α) on $C^*(G, \alpha)$ such that disintegration of representations maps a representation L of $C^*(G, \alpha)$ on \mathcal{F} to $(\varphi^u, U^u) \otimes_L \mathcal{F}$. More precisely, this representation of (G, α) lives on $C^*(G, \alpha) \otimes_L \mathcal{F}$, which we identify with \mathcal{F} by the canonical unitary $f \otimes \xi \mapsto L(f)\xi$.*

The proof also describes the universal representation (φ^u, U^u) .

Proof. We view the identity map on $C^*(G, \alpha)$ as a representation of $C^*(G, \alpha)$ on $C^*(G, \alpha)$, viewed as a Hilbert module over itself. This representation disintegrates to a representation (φ^u, U^u) of (G, α) on $C^*(G, \alpha)$. By construction, $\varphi^u: C_0(G^0) \rightarrow \mathbb{B}(C^*(G, \alpha)) = \mathcal{M}(C^*(G, \alpha))$ is the canonical morphism: a function f_0 in $C_0(G^0)$ multiplies on the left by $r^*(f_0)$ and hence on the right by $s^*(f_0)$. The unitary

$$U^u: \mathcal{L}^2(G, s, \tilde{\alpha}) \otimes_{\varphi^u} C^*(G, \alpha) \xrightarrow{\sim} \mathcal{L}^2(G, r, \alpha) \otimes_{\varphi^u} C^*(G, \alpha)$$

is the unique extension of the isomorphism

$$(6.4) \quad C_c(G^1 \times_{s, G^0, r} G^1) \xrightarrow{\sim} C_c(G^1 \times_{r, G^0, r} G^1)$$

that composes functions with the canonical homeomorphism

$$G^1 \times_{r, G^0, r} G^1 \xrightarrow{\sim} G^1 \times_{s, G^0, r} G^1, \quad (g, k) \mapsto (g, g^{-1}k).$$

A variant of Lemma 5.3 gives linear maps

$$\begin{aligned} C_c(G^1 \times_{s, G^0, r} G^1) &\rightarrow \mathcal{L}^2(G, s, \tilde{\alpha}) \otimes_{\varphi^u} C^*(G, \alpha), \\ C_c(G^1 \times_{r, G^0, r} G^1) &\rightarrow \mathcal{L}^2(G, r, \alpha) \otimes_{\varphi^u} C^*(G, \alpha) \end{aligned}$$

with dense image and shows that the isomorphism (6.4) preserves the inner products. More precisely, Lemma 5.3 considers $C_c(G^2) \odot C^*(G, \alpha)$. But we may do the same computation without the factor $C^*(G, \alpha)$, getting the above, simpler, description of U^u .

The claim that disintegration is just tensoring a given representation of $C^*(G, \alpha)$ with the universal representation is implicit in our construction in Section 5. We deduce it from the two extra properties of the bijections in Theorem 3.14, compare the proof of [7, Proposition 3.6]. The canonical unitary $C^*(G, \alpha) \otimes_L \mathcal{F} \xrightarrow{\sim} \mathcal{F}$, $f \otimes \xi \mapsto L(f)\xi$, intertwines the obvious representations of $C^*(G, \alpha)$ on both Hilbert modules. It intertwines the corresponding representations of (G, α) by 3.14.(1). The representation of (G, α) on $C^*(G, \alpha) \otimes_L \mathcal{F}$ is $(\varphi^u, U^u) \otimes_L \mathcal{F}$ by 3.14.(2). Hence the disintegration of L is obtained by transporting the representation $(\varphi^u, U^u) \otimes_L \mathcal{F}$ on $C^*(G, \alpha) \otimes_L \mathcal{F}$ to one on \mathcal{F} along the canonical unitary $C^*(G, \alpha) \otimes_L \mathcal{F} \xrightarrow{\sim} \mathcal{F}$. \square

7. TRANSFORMATION GROUPS AND ÉTALE GROUPOIDS

We now make our universal property more explicit in two cases, namely, for transformation groups and for (Hausdorff) étale groupoids. In both cases, there is a canonical Haar system α . We are going to reprove the known characterisation of $C^*(G, \alpha)$ as a crossed product for a group action in the first case and for an inverse semigroup action in the second case.

7.1. Transformation groups. Let Γ be a locally compact group and let X be a left Γ -space. Let $G = \Gamma \ltimes X$ be the transformation groupoid. Fix a Haar measure α_0 on Γ and let α be the resulting “constant” Haar system on G . By definition,

$$G^0 = X, \quad G^1 = \Gamma \times X, \quad G^2 = \Gamma^2 \times X;$$

$$s(\gamma, x) = x, \quad r(\gamma, x) = \gamma \cdot x,$$

$$d_0(\gamma_1, \gamma_2, x) = (\gamma_2, x), \quad d_1(\gamma_1, \gamma_2, x) = (\gamma_1 \gamma_2, x), \quad d_2(\gamma_1, \gamma_2, x) = (\gamma_1, \gamma_2 x).$$

The measure family $\tilde{\alpha}$ is the constant family $\tilde{\alpha}_x = \tilde{\alpha}_0 \times \delta_x$ for $x \in X$, where $\tilde{\alpha}_0$ is the right Haar measure on Γ associated to α_0 .

Let D be a C*-algebra and let \mathcal{F} be a Hilbert D -module. A representation of (G, α) consists of a representation φ of $C_0(G^0) = C_0(X)$ on \mathcal{F} and a unitary operator

$$U: \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{C_0(X)} \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(G^1, r, \alpha) \otimes_{C_0(X)} \mathcal{F}$$

with some properties.

Proposition 7.1. *A representation (φ, U) of the transformation groupoid (G, α) on \mathcal{F} is equivalent to a pair (φ, U') where U' is a representation of (Γ, α_0) such that the representation φ of $C_0(X)$ is covariant with respect to U' . Thus $C^*(G, \alpha)$ is naturally isomorphic to the crossed product $\Gamma \ltimes C_0(X)$.*

Proof. Since $G^1 = \Gamma \times X$, the Hilbert $C_0(X)$ -module $\mathcal{L}^2(G^1, s, \tilde{\alpha})$ is isomorphic to the exterior product $\mathcal{L}^2(\Gamma, \tilde{\alpha}_0) \otimes C_0(X)$; this corresponds to the constant field of Hilbert spaces over X with fibre $\mathcal{L}^2(\Gamma, \tilde{\alpha}_0)$. Hence we may simplify

$$\mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{C_0(X)} \mathcal{F} \cong \mathcal{L}^2(\Gamma, \tilde{\alpha}_0) \otimes \mathcal{F}.$$

This exterior tensor product contains $C_c(\Gamma, \mathcal{F})$ as a dense subspace. On this subspace, the representation of $C_0(G^1) = C_0(\Gamma \times X)$ acts by

$$f \cdot \xi(\gamma) = \varphi(x \mapsto f(\gamma, x))\xi(\gamma)$$

for all $f \in C_0(\Gamma \times X)$, $\xi \in C_c(\Gamma, \mathcal{F})$; that is, for fixed $\gamma \in \Gamma$, f acts by φ applied to the restricted function $x \mapsto f(\gamma, x)$. The map

$$\iota: G^1 \rightarrow G^1, \quad (\gamma, x) \mapsto (\gamma, \gamma x),$$

satisfies $s \circ \iota = r$. Hence it induces an isomorphism of Hilbert $C_0(G^0)$ -modules $\mathcal{L}^2(G^1, r, \alpha) \cong \mathcal{L}^2(G^1, s, \tilde{\alpha})$, which intertwines the standard left action of $C_0(G^1)$ on $\mathcal{L}^2(G^1, r, \alpha)$ by pointwise multiplication and the action of $C_0(G^1)$ on $\mathcal{L}^2(G^1, s, \tilde{\alpha})$ by $(f \bullet \xi)(\gamma, x) = f(\gamma, \gamma x) \cdot \xi(g)$. This induces a unitary operator

$$\mathcal{L}^2(G^1, r, \alpha) \otimes_{C_0(X)} \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(\Gamma, \alpha_0) \otimes \mathcal{F},$$

which turns the standard representation of $C_0(G)$ on $\mathcal{L}^2(G^1, r, \alpha) \otimes_{C_0(X)} \mathcal{F}$ by pointwise multiplication into the representation

$$(f \bullet \xi)(\gamma) = \varphi(x \mapsto f(\gamma, \gamma x))\xi(\gamma)$$

for all $f \in C_0(\Gamma \times X)$, $\xi \in C_c(\Gamma, \mathcal{F})$. Thus we identify the unitary operator U in a representation of (G, α) with a unitary operator

$$U': \mathcal{L}^2(\Gamma, \tilde{\alpha}_0) \otimes \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(\Gamma, \alpha_0) \otimes \mathcal{F}.$$

This intertwines the representations of $C_0(\Gamma \times X)$ specified above, and it makes a certain diagram commute. Since $C_0(\Gamma \times X) \cong C_0(\Gamma) \otimes C_0(X)$, the intertwining condition is equivalent to intertwining conditions for the representations of $C_0(\Gamma)$ and $C_0(X)$ that we get by taking functions f above that depend only on the first or second variable, respectively.

We may identify $\mathcal{L}^2(G^2, v_i, \mu_i) \otimes_{C_0(G^0)} \mathcal{F} \cong \mathcal{L}^2(\Gamma^2, \mu'_i) \otimes \mathcal{F}$, where the measure μ'_i is constructed like μ_i but for the group Γ . The commuting diagram needed for U to

be a representation is equivalent to the corresponding commuting diagram for U' . Thus U' is a representation of (Γ, α_0) if U is a representation of (G, α) . Then U' comes from a continuous representation of Γ on \mathcal{F} because $C^*(\Gamma)$ is also universal with respect to continuous representations of Γ . The condition that U' intertwines the two representations of $C_0(X)$ means exactly that the representation φ of $C_0(X)$ is covariant with respect to the continuous representation of Γ associated to U' .

The above construction may be reversed easily. So representations of (G, α) are equivalent to covariant representations for the action of Γ on $C_0(X)$. These are, in turn, equivalent to representations of the crossed product $\Gamma \ltimes C_0(X)$. So we get bijections between representations of $C^*(G, \alpha)$ and $\Gamma \ltimes C_0(X)$ on Hilbert modules over arbitrary C^* -algebras. These also satisfy the naturality properties in Theorem 3.14. Hence they come from a natural isomorphism $C^*(G, \alpha) \cong \Gamma \ltimes C_0(X)$. \square

7.2. Étale groupoids. Let G be a locally compact, Hausdorff, étale groupoid. Endow G with the canonical Haar system α where each α^x is the counting measure on the discrete subset $G^x \subseteq G^1$. Then $\tilde{\alpha}$ is the family of counting measures $\tilde{\alpha}_x$ on the discrete subsets $G_x \subseteq G^1$. Similarly, all the other measures λ_i and μ_i associated to the maps d_i and v_i for $i = 0, 1, 2$ are counting measures.

We quickly recall the relationship between étale groupoids and inverse semigroup actions on spaces, see [4, 10]. A *bisection* of G is an open subset $a \subseteq G^1$ such that the restrictions of the source and range maps

$$s_a: a \rightarrow s(a), \quad r_a: a \rightarrow r(a),$$

are injective or, equivalently, homeomorphisms onto their images. The bisections in G form an inverse semigroup $\text{Bis}(G)$. It acts on G^0 by the partial homeomorphisms $\theta_a = r_a \circ s_a^{-1}: s(a) \xrightarrow{\sim} r(a)$. The transformation groupoid $\text{Bis}(G) \ltimes G^0$ for this action is naturally isomorphic to G (see also [5]). Here we may replace $\text{Bis}(G)$ by any inverse subsemigroup $S \subseteq \text{Bis}(G)$ that is *wide*, that is,

$$(7.2) \quad \bigcup_{t \in S} t = G^1, \quad \bigcup_{v \in S, v \leq u, t} v = u \cap t \quad \text{for all } u, t \in S.$$

These two conditions say that the canonical groupoid homomorphism $S \ltimes G^0 \rightarrow \text{Bis}(G) \ltimes G^0 \cong G^1$ induced by the inclusion $S \rightarrow \text{Bis}(G)$ is bijective. Then it is a homeomorphism because it is always a local homeomorphism. That is, $G \cong S \ltimes G^0$ as topological groupoids if and only if S is wide. We fix an inverse semigroup action θ of an inverse semigroup S on a locally compact space $X = G^0$ and an isomorphism $G \cong S \ltimes_\theta X$. The action of S on X induces an action on the C^* -algebra $C_0(X)$ by partial isomorphisms. This action has a crossed product $S \ltimes C_0(X)$. It is already known that $C^*(G) \cong S \ltimes C_0(X)$, see [4, Theorem 9.8], for instance. The same result is also proved in [1, 2, 10, 11], sometimes under mild extra conditions. Our universal property implies another proof of this fact, assuming G to be Hausdorff. Representations of the groupoid G may be simplified, using that G is étale. As a result, these representations are equivalent to “covariant representations” of the S -action on $C_0(X)$. Finally, these covariant representations are the same as representations of the crossed product $S \ltimes C_0(X)$. Thus $C^*(G)$ and $S \ltimes C_0(X)$ have equivalent representations on arbitrary Hilbert modules. This implies that they are isomorphic. The following definition describes covariant representations of the S -action on $C_0(X)$, which is the only case we need.

Definition 7.3. Let X be a locally compact Hausdorff space endowed with an action of an inverse semigroup S by partial homeomorphisms $\theta_a: D_{a^*a} \xrightarrow{\sim} D_{aa^*}$ between open subsets $D_e \subseteq X$ for $e \in E(S)$. A *covariant representation* of this system on a Hilbert D -module \mathcal{F} consists of a nondegenerate representation

$\varphi: C_0(X) \rightarrow \mathbb{B}(\mathcal{F})$ and a family of unitaries $U_a: \mathcal{F}_{a^*a} \xrightarrow{\sim} \mathcal{F}_{aa^*}$ for $a \in S$, where $\mathcal{F}_e := \varphi(C_0(D_e))\mathcal{F}$, that satisfy the following conditions:

- (1) U_b restricts to U_a if $a \leq b$ in S ;
- (2) $U_a^* = U_{a^*}$ for all $a \in S$;
- (3) $U_a U_b = U_{ab}$ if $a, b \in S$ with $a^*a = bb^*$;
- (4) $U_a \varphi(f) U_a^* = \varphi(f \circ \theta_{a^*})$ for all $f \in C_0(D_{a^*a})$.

If S is a group, so that $1 \in S$ is the only idempotent element, then it is very well known that covariant representations in the sense above are equivalent to representations of the crossed product $S \ltimes C_0(X)$. For inverse semigroups, it seems that covariant representations have so far only been introduced on Hilbert spaces; the standard reference for this is [15]. Hilbert module representations require extra care because the Hilbert submodules \mathcal{F}_e need not be complementable. For instance, let S be a semilattice of open subsets D_e of X with the intersection as product, acting on X by identity maps $\theta_e: D_e \xrightarrow{\sim} D_e$. The identity map $\varphi: C_0(X) \xrightarrow{\sim} C_0(X)$ and the identity maps $U_e: C_0(D_e) \xrightarrow{\sim} C_0(D_e)$ form a covariant representation of this action on $\mathcal{F} = C_0(X)$ viewed as a Hilbert $C_0(X)$ -module. The submodule $\mathcal{F}_e = C_0(D_e)$ is only complementable in \mathcal{F} if D_e is clopen. Hence we cannot replace the partially defined maps U_a above by partial isometries on \mathcal{F} . The following proposition asserts that our notion of covariant representation, when specialised to Hilbert spaces, is equivalent to Sieben's [15, Definition 3.4].

Proposition 7.4. *Let (X, S, θ) be an action of S on X and let \mathcal{H} be a Hilbert space. Then a covariant representation of (X, θ, S) on \mathcal{H} is equivalent to a nondegenerate representation $\varphi: C_0(X) \rightarrow \mathbb{B}(\mathcal{H})$ together with a map $S \ni a \mapsto U_a \in \mathbb{B}(\mathcal{H})$ with $U_e(\mathcal{H}) = \varphi(D_e)\mathcal{H}$ for all $e \in E(S)$ and $U_a^* = U_{a^*}$, $U_{ab} = U_a U_b$ and $\varphi(f \circ \theta_{a^*}) = U_a \varphi(f) U_a^*$ for all $a, b \in S$ and $f \in C_0(D_{a^*a}) \subseteq C_0(X)$.*

Proof. Let φ and $(U_a)_{a \in S}$ be as in the statement. Each U_a is a partial isometry of \mathcal{H} with source projection $U_a^* U_a = U_{a^*a}$, the orthogonal projection onto $\mathcal{H}_{a^*a} := \varphi(C_0(D_{a^*a}))\mathcal{H}$, and range projection $U_a U_a^*$, the orthogonal projection onto \mathcal{H}_{aa^*} . Hence we may view U_a as an isomorphism $\mathcal{H}_{a^*a} \xrightarrow{\sim} \mathcal{H}_{aa^*}$. In this way, φ and the isomorphisms $U_a: \mathcal{H}_{a^*a} \xrightarrow{\sim} \mathcal{H}_{aa^*}$ form a covariant representation as in Definition 7.3. Conversely, let $\varphi: C_0(X) \rightarrow \mathbb{B}(\mathcal{H})$ and $U_a: \mathcal{H}_{a^*a} \xrightarrow{\sim} \mathcal{H}_{aa^*}$ for $a \in S$ form a covariant representation of (X, S, θ) as in Definition 7.3, where $\mathcal{H}_e := \varphi(C_0(D_e))\mathcal{H}$. Extend U_a by zero on the orthogonal complement of \mathcal{H}_{a^*a} to view it as a partial isometry $U_a \in \mathbb{B}(\mathcal{H})$. It remains to show that $U_a U_b = U_{ab}$ for all $a, b \in S$. We already have this relation if $a^*a = bb^*$. And we have assumed that U_b restricts to U_a if $a \leq b$.

Let $e, f \in E(S)$. Then $\mathcal{H}_{ef} \subseteq \mathcal{H}_e \cap \mathcal{H}_f$ is obvious. Conversely, if $\xi \in \mathcal{H}_e \cap \mathcal{H}_f$, then $\varphi(u_i^e)\xi \rightarrow \xi$ for an approximate unit (u_i^e) of $C_0(D_e)$ and $\varphi(u_i^f)\xi \rightarrow \xi$ for an approximate unit (u_i^f) of $C_0(D_f)$. Since $u_i^e \cdot u_i^f \in C_0(D_{ef})$, this implies $\xi \in \mathcal{H}_{ef}$. Thus $\mathcal{H}_e \cap \mathcal{H}_f = \mathcal{H}_{ef}$. This is equivalent to $U_e U_f = U_{ef}$. Using this, the assumption that U_b restricts to U_a if $a \leq b$ implies $U_a U_e = U_{ae}$ for all $a \in S, e \in E(S)$. Thus

$$U_a U_b = U_a U_{a^*a} U_{bb^*} U_b = U_a U_{bb^*} U_{a^*a} U_b = U_{abb^*} U_{a^*ab} = U_{abb^*a^*ab} = U_{ab}$$

for all $a, b \in S$ because $a(a^*a) = a$, $b(b^*b) = b$ and $(abb^*)(a^*ab) = ab$ with matching range and source projections. \square

Theorem 7.5. *There are bijections between representations of $S \ltimes C_0(X)$ and covariant representations of (X, S, θ) on Hilbert modules \mathcal{F} over arbitrary C*-algebras D . These have the naturality properties in Theorem 3.14.*

Proof. For Hilbert space representations, this follows from Proposition 7.4 and the definition of the crossed product, compare [15]. The assertion for general \mathcal{F} may

be proved in a similar fashion as follows. The crossed product $S \ltimes C_0(X)$ can be defined as the universal C^* -algebra generated by expressions of the form $f_a \delta_a$ with $a \in S$ and $f_a \in C_0(D_{aa^*})$, subject to the relations that $f_a \mapsto f_a \delta_a$ is a linear map $C_0(D_{aa^*}) \rightarrow S \ltimes C_0(X)$, that $f \delta_a = f \delta_b$ if $a \leq b$ in S and $f \in C_0(D_{aa^*})$, plus the following algebraic relations:

$$(f_a \delta_a) \cdot (f_b \delta_b) = ((f_a \circ \theta_a) \cdot f_b) \circ \theta_{a^*} \delta_{ab}, \quad (f_a \delta_a)^* = (\overline{f}_a \circ \theta_a) \delta_{a^*}$$

for all $a, b \in S$, $f_a \in C_0(D_{aa^*})$ and $f_b \in D_{bb^*}$. A covariant representation $(\varphi, (U_a)_{a \in S})$ of (X, S, θ) on a Hilbert module \mathcal{F} integrates to a representation $U \ltimes \varphi: S \ltimes C_0(X) \rightarrow \mathbb{B}(\mathcal{F})$ by $U \ltimes \varphi(f_a \delta_a) := \varphi(f_a) U_a$. The covariance conditions imply that this is a well defined, nondegenerate $*$ -homomorphism. Conversely, given a representation $\rho: S \ltimes C_0(X) \rightarrow \mathbb{B}(\mathcal{F})$, we define $\varphi: C_0(X) \rightarrow \mathbb{B}(\mathcal{F})$ by $\varphi(f) := \rho(f \delta_1)$, where $1 \in S$ denotes the unit of S ; for simplicity we assume here that S is unital. This is no loss of generality because we can always add a formal unit to S and extend the action to the unitisation without changing the crossed product. Then φ is a nondegenerate $*$ -homomorphism. Given $a \in S$, we define $U_a: \mathcal{F}_{a^*a} \rightarrow \mathcal{F}_{aa^*}$ by $U_a(\rho(f \delta_{a^*a}) \xi) := \rho((f \circ \theta_{a^*}) \delta_a) \xi$ for all $f \in C_0(D_{a^*a})$ and $\xi \in \mathcal{F}$. By definition, \mathcal{F}_{a^*a} consists of elements of the form $\rho(f \delta_{a^*a}) \xi = \varphi(f) \xi$ with $f \in C_0(D_{a^*a})$ and $\xi \in \mathcal{F}$. Writing f as a product of two elements of $C_0(D_{a^*a})$ and using the definition of φ , we get $\rho((f \circ \theta_{a^*}) \delta_a) \xi \in \mathcal{F}_{aa^*}$. The relation

$$((f_1 \circ \theta_{a^*}) \delta_a)^* \cdot (f_2 \circ \theta_{a^*}) \delta_a = (\overline{f}_1 \cdot f_2) \delta_{a^*a}$$

holds in $S \ltimes C_0(X)$ for all $f_1, f_2 \in C_0(D_{a^*a})$. Hence the map U_a is a well defined isometry $\mathcal{F}_{a^*a} \rightarrow \mathcal{F}_{aa^*}$:

$$\begin{aligned} \langle \rho((f_1 \circ \theta_{a^*}) \delta_a) \xi_1 | \rho((f_2 \circ \theta_{a^*}) \delta_a) \xi_2 \rangle &= \langle \xi_1 | \rho(((f_1 \circ \theta_{a^*}) \delta_a)^* \cdot (f_2 \circ \theta_{a^*}) \delta_a) \xi_2 \rangle \\ &= \langle \xi_1 | \rho(\overline{f}_1 \cdot f_2) \delta_{a^*a} \xi_2 \rangle = \langle \rho(f_1 \delta_{a^*a}) \xi_1 | \rho(f_2 \delta_{a^*a}) \xi_2 \rangle. \end{aligned}$$

By definition, the image of U_a is $\rho(C_0(D_{aa^*}) \delta_a) \mathcal{F} \subseteq \mathcal{F}_{aa^*} = \varphi(C_0(D_{aa^*})) \mathcal{F} = \rho(C_0(D_{aa^*}) \delta_{aa^*}) \mathcal{F}$. Indeed, this inclusion is an equality so that U_a is unitary; the other inclusion follows because $C_0(D_{aa^*}) \delta_{aa^*} = (C_0(D_{aa^*}) \delta_a) \cdot (C_0(D_{a^*a}) \delta_{a^*})$ in $S \ltimes C_0(X)$. By construction, U_a satisfies $U_a \varphi(f) U_{a^*} = \varphi(f \circ \theta_{a^*})$ for all $f \in C_0(D_{a^*a})$. Clearly, $U_a^* = U_{a^*}$ and U_a restricts to U_b if $b \leq a$. The remaining multiplicativity property $U_a U_b = U_{ab}$ for $a^*a = bb^*$ is also easily checked. Therefore, φ and the partial unitaries U_a for $a \in S$ form a covariant representation of (X, S, θ) . By construction, $U \ltimes \varphi = \rho$. Thus $(\varphi, U) \mapsto U \ltimes \varphi$ implements the desired bijection between covariant representations and representations of the crossed product $S \ltimes C_0(X)$. We leave it to the reader to check that these bijections have the naturality properties in Theorem 3.14. \square

Theorem 7.6. *Let S be an inverse semigroup and let X be a locally compact space with an action θ of S by partial homeomorphisms. Let $G := S \ltimes X$. Assume that this étale, locally compact groupoid is Hausdorff. Let \mathcal{F} be a Hilbert D -module. Representations of G on \mathcal{F} correspond bijectively to covariant representations of (X, S, θ) on \mathcal{F} , which correspond bijectively to representations of $S \ltimes_\theta C_0(X)$ on \mathcal{F} . These bijections have the naturality properties in Theorem 3.14. So they come from a unique isomorphism $C^*(G) \cong S \ltimes_\theta C_0(X)$.*

Proof. First we construct a covariant representation of (X, S, θ) on \mathcal{F} from a representation (U, φ) of $S \ltimes X$ on \mathcal{F} . Here $\varphi: C_0(G^0) \rightarrow \mathbb{B}(\mathcal{F})$ is a nondegenerate representation and U is an isomorphism of $C_0(G^1)$ - D -correspondences

$$U: \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_\varphi \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(G^1, r, \alpha) \otimes_\varphi \mathcal{F}$$

with $d_2^*(U) d_0^*(U) = d_1^*(U)$. For an open subset $e \subseteq G^0$, let $\mathcal{F}_e := \varphi(C_0(e)) \cdot \mathcal{F}$. Any $a \in S$ gives a bisection of G^1 , namely, the set of all germs of pairs (a, x)

with $x \in D_{a^*a}$. By abuse of notation, we also denote this bisection by a . The isomorphism U restricts to an isomorphism of $C_0(a)$ - D -correspondences

$$U|_a: C_0(a) \cdot \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F} \xrightarrow{\sim} C_0(a) \cdot \mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F}.$$

There are canonical isomorphisms of $C_0(a)$ - $C_0(G^0)$ -correspondences

$$C_0(a) \cdot \mathcal{L}^2(G^1, s, \tilde{\alpha}) \cong C_0(s(a)), \quad C_0(a) \cdot \mathcal{L}^2(G^1, r, \alpha) \cong C_0(r(a)).$$

The first isomorphism sends a function $\xi \in C_0(a) \cdot C_c(G^1) = C_c(a)$ to $\xi \circ s_a^{-1} \in C_c(s(a))$, and similarly for the second. Here we view the ideal $C_0(s(a))$ in $C_0(G^0)$ as a $C_0(a)$ - $C_0(G^0)$ -correspondence in the canonical way, using the homeomorphism s_a . This gives canonical isomorphisms of $C_0(a)$ - D -correspondences

$$\begin{aligned} C_0(a) \cdot \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{\varphi} \mathcal{F} &\cong C_0(s(a)) \otimes_{\varphi} \mathcal{F} \cong \varphi(C_0(s(a)))\mathcal{F} = \mathcal{F}_{s(a)}, \\ C_0(a) \cdot \mathcal{L}^2(G^1, r, \alpha) \otimes_{\varphi} \mathcal{F} &\cong C_0(r(a)) \otimes_{\varphi} \mathcal{F} \cong \varphi(C_0(r(a)))\mathcal{F} = \mathcal{F}_{r(a)}. \end{aligned}$$

Thus $U|_a$ becomes an isomorphism of $C_0(a)$ - D -correspondences

$$U_a: \mathcal{F}_{s(a)} \xrightarrow{\sim} \mathcal{F}_{r(a)},$$

where we view $\mathcal{F}_{s(a)}$ and $\mathcal{F}_{r(a)}$ as $C_0(a)$ - D -correspondences using the homeomorphisms s_a and r_a . That is,

$$U_a(\varphi(f \circ s_a^{-1})\xi) = \varphi(f \circ r_a^{-1})U_a(\xi) \quad \text{for all } f \in C_0(a), \xi \in \mathcal{F}_{s(a)}.$$

We reinterpret this as a covariance condition. The inverse semigroup S acts on the C*-algebra $C_0(G^0)$ by the isomorphisms

$$\gamma_a: C_0(s(a)) \xrightarrow{\sim} C_0(r(a)), \quad f \mapsto f \circ \theta_{a^*} = f \circ s_a \circ r_a^{-1}.$$

Thus $U_a \varphi(f) U_a^* = \varphi(\gamma_a(f))$ for all $f \in C_0(s(a))$ as in Definition 7.3.(4). The conditions in (1) and (2) in Definition 7.3 are also clear from our construction. We claim that the condition $d_2^*(U) \circ d_0^*(U) = d_1^*(U)$ for U to be a representation implies

$$(7.7) \quad U_a \circ U_b = U_{ab} \quad \text{for all } a, b \in S \text{ with } a^*a = bb^*.$$

Fix $a, b \in S$ with $a^*a = bb^*$ as above. Let $t := a \times_{s,r} b \subseteq G^2$ be the set of all pairs $(g, h) \in G^1 \times G^1$ with $g \in a$, $h \in b$ and $s(g) = r(h)$. This is an open subset of G^2 , and all three vertex maps $v_i: G^2 \rightarrow G^0$, $i = 0, 1, 2$, are injective on t because a and b are bisections of G . We call subsets of G^2 with these properties *trisections*. Since $a^*a = bb^*$, we have

$$v_0(t) = r(a) = r(ab), \quad v_1(t) = s(a) = r(b), \quad v_2(t) = s(b) = s(ab).$$

The face maps $d_i: G^2 \rightarrow G^1$ for $i = 0, 1, 2$, are also injective on t , and

$$d_0(t) = a, \quad d_1(t) = ab, \quad d_2(t) = b.$$

The pullback $d_i^*(U): \mathcal{L}^2(v_j) \otimes_{C_0(G^0)} \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(v_k) \otimes_{C_0(G^0)} \mathcal{F}$ for $i \in \{0, 1, 2\}$ and appropriate $j, k \in \{0, 1, 2\}$ depending on i is defined as the map

$$\text{id} \otimes U: \mathcal{L}^2(d_i) \otimes_{C_0(G^1)} \mathcal{L}^2(s) \otimes_{C_0(G^0)} \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(d_i) \otimes_{C_0(G^1)} \mathcal{L}^2(r) \otimes_{C_0(G^0)} \mathcal{F}$$

composed with the canonical isomorphisms

$$\mathcal{L}^2(d_i) \otimes_{C_0(G^1)} \mathcal{L}^2(s) \cong \mathcal{L}^2(v_j), \quad \mathcal{L}^2(d_i) \otimes_{C_0(G^1)} \mathcal{L}^2(r) \cong \mathcal{L}^2(v_k).$$

Therefore, the restriction of $d_i^*(U)$ to $t \subseteq G^2$ corresponds to the restriction of $\text{id} \otimes U$ to t . As before, this gives an isomorphism

$$\begin{aligned} C_0(d_i(t)) \cdot \mathcal{L}^2(s) \otimes_{C_0(G^0)} \mathcal{F} &\cong C_0(t) \cdot \mathcal{L}^2(d_i) \otimes_{C_0(G^1)} \mathcal{L}^2(s) \otimes_{C_0(G^0)} \mathcal{F} \\ &\xrightarrow{\sim} C_0(t) \cdot \mathcal{L}^2(d_i) \otimes_{C_0(G^1)} \mathcal{L}^2(r) \otimes_{C_0(G^0)} \mathcal{F} \cong C_0(d_i(t)) \cdot \mathcal{L}^2(r) \otimes_{C_0(G^0)} \mathcal{F}. \end{aligned}$$

It coincides with the restriction of U to $d_i(t)$. The equality $d_2^*(U) \circ d_0^*(U) = d_1^*(U)$ implies an equality for all these restrictions; in particular, it implies $d_2^*(U)|_t \circ d_0^*(U)|_t =$

$d_1^*(U)|_t$. This is equivalent to $U_a \circ U_b = U_{ab}$ by the above identifications. Thus φ and the maps U_a for $a \in S$ form a representation of (X, S, θ) if (φ, U) is a representation of G .

Next we reverse this construction, showing that any covariant representation $\varphi, (U_a)_{a \in S}$ comes from a unique representation (φ, U) . We continue to identify elements of S with the corresponding bisections in G , which are open subsets of G . First, we claim that the restrictions of U_a and U_b to $\mathcal{F}_{s(a \cap b)}$ are equal for all $a, b \in S$. Definition 7.3.(1) implies that U_a and U_b coincide on $\mathcal{F}_{s(c)}$ for each $c \in S$ with $c \leq a, b$. Since S is a wide inverse semigroup in $\text{Bis}(G)$, $a \cap b$ is the union of the bisections $c \in S$ with $c \leq a, b$. Thus $\mathcal{F}_{s(a \cap b)} = \sum_{c \leq a, b} \mathcal{F}_{s(c)}$, and we get the claim.

Extending a function on $a \subseteq G^1$ by 0 and summing, we map $\bigoplus_{a \in S} C_c(a)$ to $C_c(G^1)$. A partition of unity argument shows that this map to $C_c(G^1)$ is surjective. We define similar maps

$$\begin{aligned} \tau_s: \bigoplus_{a \in S} C_c(a) \odot \mathcal{F}_{s(a)} &\rightarrow \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{C_0(G^0)} \mathcal{F}, \\ \tau_r: \bigoplus_{a \in S} C_c(a) \odot \mathcal{F}_{r(a)} &\rightarrow \mathcal{L}^2(G^1, r, \alpha) \otimes_{C_0(G^0)} \mathcal{F}. \end{aligned}$$

We claim that both have dense range. If we replaced $\mathcal{F}_{s(a)}$ and $\mathcal{F}_{r(a)}$ by \mathcal{F} , this would follow from the density of $C_c(G^1) \odot \mathcal{F}$ in the right hand sides. Since $C_c(a) = C_c(a) \cdot C_c(a)$, we may rewrite the image of $f \otimes \xi$ for $f \in C_c(a)$, $\xi \in \mathcal{F}$ as $f_1 \cdot f_2 \otimes \xi \equiv f_1 \otimes f_2 \cdot \xi$ with $f_1, f_2 \in C_c(a)$ and hence $f_2 \cdot \xi \in \mathcal{F}_{s(a)}$ when we work in $\mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{C_0(G^0)} \mathcal{F}$ and $f_2 \cdot \xi \in \mathcal{F}_{r(a)}$ when we work in $\mathcal{L}^2(G^1, r, \alpha) \otimes_{C_0(G^0)} \mathcal{F}$. The maps $(U_a)_{a \in S}$ give an isomorphism

$$U: \bigoplus_{a \in S} C_c(a) \odot \mathcal{F}_{s(a)} \xrightarrow{\sim} \bigoplus_{a \in S} C_c(a) \odot \mathcal{F}_{r(a)}, \quad (f_a \otimes \xi_a)_{a \in S} \mapsto (f_a \otimes U_a(\xi_a))_{a \in S}.$$

We claim that $\langle \tau_r \circ U(x) | \tau_r \circ U(y) \rangle = \langle \tau_s(x) | \tau_s(y) \rangle$ for all $x, y \in \bigoplus_{a \in S} C_c(a) \odot \mathcal{F}_{s(a)}$. Hence there is a unique unitary operator

$$\bar{U}: \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{C_0(G^0)} \mathcal{F} \xrightarrow{\sim} \mathcal{L}^2(G^1, r, \alpha) \otimes_{C_0(G^0)} \mathcal{F}$$

with $\bar{U} \circ \tau_s = \tau_r \circ U$. It suffices to prove the claim if $x = f_1 \otimes \xi_1$ and $y = f_2 \otimes \xi_2$ with $f_1 \in C_c(a)$, $\xi_1 \in \mathcal{F}_{s(a)}$, $f_2 \in C_c(b)$, $\xi_2 \in \mathcal{F}_{s(b)}$ for some $a, b \in S$. So we must prove that

$$\langle \xi_1 | \langle f_1 | f_2 \rangle_s \xi_2 \rangle = \langle U_a(\xi_1) | \langle f_1 | f_2 \rangle_r U_b(\xi_2) \rangle.$$

The function $\langle f_1 | f_2 \rangle_s$ is simply the function $s(x) \mapsto f_1(x) \cdot \overline{f_2(x)}$ for $x \in a \cap b$, extended by 0 outside $s(a \cap b)$, and $\langle f_1 | f_2 \rangle_r = \langle f_1 | f_2 \rangle_s \circ \theta_a^*$. We may rewrite $\langle f_1 | f_2 \rangle_s = \overline{f_3} \cdot f_4$ with $f_3, f_4 \in C_c(s(a \cap b))$. Hence $\langle \xi_1 | \langle f_1 | f_2 \rangle_s \xi_2 \rangle = \langle f_3 \xi_1 | f_4 \xi_2 \rangle$ and

$$\begin{aligned} \langle U_a(\xi_1) | \langle f_1 | f_2 \rangle_r U_b(\xi_2) \rangle &= \langle U_a(\xi_1) | (\langle f_1 | f_2 \rangle_s \circ \theta_b^*) U_b(\xi_2) \rangle \\ &= \langle (f_3 \circ \theta_a^*) U_a(\xi_1) | (f_4 \circ \theta_b^*) U_b(\xi_2) \rangle \\ &= \langle U_a(f_3 \xi_1) | U_b(f_4 \xi_2) \rangle = \langle U_a(f_3 \xi_1) | U_a(f_4 \xi_2) \rangle = \langle f_3 \xi_1 | f_4 \xi_2 \rangle. \end{aligned}$$

Here we use that θ_a and θ_b agree on $s(a \cap b)$ and hence on the supports of f_3, f_4 , that U_a and U_b agree on $\mathcal{F}_{s(a \cap b)}$ and are unitary. The computation above proves our claim that $\langle \tau_r \circ U(x) | \tau_r \circ U(y) \rangle = \langle \tau_s(x) | \tau_s(y) \rangle$ for all $x, y \in \bigoplus_{a \in S} C_c(a) \odot \mathcal{F}_{s(a)}$. This finishes the construction of the unitary \bar{U} .

The unitary \bar{U} is the only one that acts by U_a on $C_c(a) \cdot \mathcal{L}^2(G^1, s, \tilde{\alpha}) \otimes_{C_0(X)} \mathcal{F}$ because the latter is the closure of the τ_s -image of $C_c(a) \otimes \mathcal{F}_{s(a)}$. The covariance condition (4) in Definition 7.3 for the unitaries U_a says that \bar{U} intertwines the left actions of $C_0(G^1)$, that is, it is an isomorphism of correspondences. Since the trisections described above cover G^2 , the equality $d_2^*(U) \circ d_0^*(U) = d_1^*(U)$ follows

from 7.7 by reversing the computation above. Thus \bar{U} is a representation of G . This finishes the proof of the bijection between representations of G and covariant representations of (X, S, θ) .

Finally, we check that our bijection between representations of G and covariant representations of $(C_0(X), S, \theta)$ also satisfies the naturality properties in Theorem 3.14. Let $V: \mathcal{F} \hookrightarrow \mathcal{F}'$ be a Hilbert module isometry. Let \mathcal{F} and \mathcal{F}' carry representations (φ, U) and (φ', U') of G . If V intertwines these representations, then it maps \mathcal{F}_{a^*a} into \mathcal{F}'_{a^*a} for all $a \in S$, and the restricted isometries $\mathcal{F}_{a^*a} \hookrightarrow \mathcal{F}'_{a^*a}$ and $\mathcal{F}_{aa^*} \hookrightarrow \mathcal{F}'_{aa^*}$ intertwine the unitaries $U_a: \mathcal{F}_{a^*a} \xrightarrow{\sim} \mathcal{F}_{aa^*}$ and $U'_a: \mathcal{F}'_{a^*a} \rightarrow \mathcal{F}'_{aa^*}$. Thus V intertwines the covariant representations of (X, S, θ) associated to (φ, U) . Conversely, if V intertwines φ and $(U_a)_{a \in S}$ and the corresponding family φ' and $(U'_a)_{a \in S}$, then it must intertwine U and U' because we may reconstruct U from $(U_a)_{a \in S}$ as above. Thus our bijection has the first naturality property in Theorem 3.14. The second naturality property is also routine to check. Since the bijection between representations of $C^*(G)$ and $S \rtimes_\theta C_0(X)$ has these two naturality properties, it is induced by an isomorphism $C^*(G) \cong S \rtimes_\theta C_0(X)$. \square

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